Physics 322 Solution to Homework Set #5 Spring 2008

1. Problem 6.16 in your textbook. By symmetry, $\vec{B}$ and $\vec{H}$ will be in the $\hat{\phi}$ direction. Using Ampere's law:

$$\int_C \vec{H}\cdot d\vec{l} = I_{free,enc} = I = 2\pi sH \quad => \quad \vec{H} = \frac{I}{2\pi s}\hat{\phi}$$

$$\vec{B} = \mu\vec{H} = \mu_0(1+\chi_m)\vec{H} = \mu_0(1+\chi_m)\frac{I}{2\pi s}\hat{\phi}; \qquad \vec{M} = \chi_m\vec{H} = \frac{\chi_m I}{2\pi s}\hat{\phi}$$

$$\vec{J}_b = \vec{\nabla}\times\vec{M} = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\chi_m I}{2\pi s}\right)\hat{z} = 0 \qquad \vec{K}_b = \vec{M}\times\hat{n} = \frac{\chi_m I}{2\pi a}\hat{z} \text{ at } s=a \text{ and } \frac{-\chi_m I}{2\pi b}\hat{z} \text{ at } s=b$$

The total enclosed current for an amperian loop between a and b is then $I + 2\pi a K_b(a)$

$$I_{enc} = I + \frac{\chi_m I}{2\pi a}(2\pi a) = (1+\chi_m)I \quad => \quad \int_C \vec{B}\cdot d\vec{l} = \mu_0 I_{enc} = \mu_0(1+\chi_m)I = 2\pi sB$$

$$=> \quad \vec{B} = \mu_0(1+\chi_m)\frac{I}{2\pi s}\hat{\phi}$$

2. Problem 6.19 in your textbook. For $\Delta\vec{m}$ being the change in magnetic dipole moment per atom, the magnetization is $\vec{M} = \Delta\vec{m}/V$ where V is the volume of an atom. From Eq. 6.8,

$$\Delta\vec{m} = -\frac{e^2r^2}{4m_e}\vec{B} \quad => \quad \vec{M} = -\frac{e^2r^2}{4m_eV}\vec{B} = \chi_m\vec{H} = \frac{\chi_m}{\mu}\vec{B} = \frac{\chi_m}{\mu_0(1+\chi_m)}\vec{B}$$

For $\chi_m << 1$ as it is in diamagnetic materials, we can take $1+\chi_m \approx 1$ giving us:

$$\chi_m = -\frac{e^2r^2\mu_0}{4m_eV} \qquad \text{For} \quad V = \frac{4}{3}\pi r^3, \quad \chi_m = -\frac{\mu_0}{4\pi}\frac{3e^2}{4m_er}$$

Taking $r = 10^{-10}\ m$ (1 Angstrom),

$$\chi_m = -(10^{-7})\frac{3(1.6\times10^{-19})^2}{4(9.1\times10^{-31})(10^{-10})} = 2\times10^{-5}$$

This estimate is the right order of magnitude (cf -1×10^{-5} in Table 6.1). It assumes one orbital electron per atom, while copper has two, it assumes the same radius for the orbital electrons and volume factor, and uses a crude estimate for both radii.

3. Problem 6.24 in your textbook. Use the result that the electric field of a uniformly charged sphere of radius R is:

$$\vec{E}_{in} = \frac{\rho r}{3\epsilon_0}\hat{r} \qquad \vec{E}_{out} = \frac{\rho R^3}{3\epsilon_0 r^2}\hat{r}$$

$$Eq.2.15: \quad \vec{E}(\vec{r}) = \rho\left[\frac{1}{4\pi\epsilon_0}\int_V \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\,d\tau'\right] \qquad \text{for uniform charge density}$$

$$Eq.4.9: \quad V(\vec{r}) = \vec{P}\cdot\left[\frac{1}{4\pi\epsilon_0}\int_V \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\,d\tau'\right] \qquad \text{for uniform polarization}$$

$$Eq.6.11: \quad \vec{A}(\vec{r}) = \mu_0\epsilon_0\vec{M}\times\left[\frac{1}{4\pi\epsilon_0}\int_V \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\,d\tau'\right] \qquad \text{for uniform magnetization}$$

Given $\vec{E}$ for a uniformly polarized sphere, above, we can write down the potential from a uniformly polarized sphere:

$$\rho\vec{r} \rightarrow \vec{P}\cdot\vec{r} \quad => \quad V_{in} \;=\; \frac{1}{3\epsilon_0}\vec{P}\cdot\vec{r} \qquad V_{out} \;=\; \frac{R^3}{3\epsilon_0 r^2}\vec{P}\cdot\hat{r}$$

Similarly, we can write down the vector potential from a uniformly magnetized sphere:

$$\rho\vec{r} \rightarrow \mu_0\epsilon_0\vec{M}\times\vec{r} \quad => \quad \vec{A}_{in} \;=\; \frac{\mu_0}{3}\vec{M}\times\vec{r} \quad \text{and} \quad \vec{A}_{out} \;=\; \frac{\mu_0 R^3}{3r^2}\vec{M}\times\vec{r}$$

4. Problem 6.25 in your textbook.

(a) From Eq. 5.86 with $\theta = 0$ (on axis), the magnetic field from the lower magnet is:

$$\vec{B}_1(z) \;=\; \frac{\mu_0}{4\pi}\,\frac{2m}{z^3}\hat{z} \quad => \quad \vec{m}_2\cdot\vec{B}_1 \;=\; -\frac{\mu_0}{2\pi}\,\frac{2m^2}{z^3}$$

because the moments of the two magnets are opposite. The magnetic force on the upper magnet is:

$$\vec{F}_2 \;=\; \vec{\nabla}(\vec{m}_2\cdot\vec{B}_1) \;=\; \frac{\partial}{\partial z}\left[-\frac{\mu_0}{2\pi}\,\frac{2m^2}{z^3}\right]\hat{z} \;=\; \frac{3\mu_0 m^2}{2\pi z^4}\;\hat{z}$$

This upward magnetic force balances the downward gravitational force ($-m_d g\hat{z}$), when

$$\frac{3\mu_0 m^2}{2\pi z^4} \;=\; m_d g \quad => \quad z \;=\; \left[\frac{3\mu_0 m^2}{2\pi m_d g}\right]^{1/4}$$

(b) Let z be the distance from the lower magnet to the middle magnet and y be the distance from the middle magnet to the upper magnet. The middle magnet is repelled upward by the loer magnet and pushed downward by the upper magnet. Therefore:

$$\frac{3\mu_0 m^2}{2\pi z^4} - \frac{3\mu_0 m^2}{2\pi y^4} - m_d g \;=\; 0$$

The upper magnet is repelled upward by the middle magnet and attracted downward by the lower magnet. Therefore:

$$\frac{3\mu_0 m^2}{2\pi y^4} - \frac{3\mu_0 m^2}{2\pi (z+y)^4} - m_d g \;=\; 0$$

Subtracting the two equations,

$$\frac{3\mu_0 m^2}{2\pi}\left[\frac{1}{z^4} - \frac{1}{y^4} - \frac{1}{y^4} + \frac{1}{(z+y)^4}\right] - m_d g + m_d g \;=\; 0 \quad => \quad \frac{1}{z^4} - \frac{2}{y^4} + \frac{1}{(z+y)^4} \;=\; 0$$

Letting $a = z/y$, we have:

$$2 \;=\; \frac{1}{(z/y)^4} + \frac{1}{(z/y+1)^4} \quad => \quad 2 \;=\; \frac{1}{a^4} + \frac{1}{(a+1)^4}$$

Using mathmatica, I get $a = z/y = .850$.

5. Problem 6.26 in your textbook. At the interface, the perpendicular component of $\vec{B}$ is continuous as is the parallel component of $\vec{H}$, and, as always, $\vec{B} = \mu \vec{H}$. Therefore,

$$B_1^{\perp} \;=\; B_2^{\perp} \quad \text{and} \quad H_1^{||} \;=\; H_2^{||} \quad => \quad \frac{1}{\mu_1} B_1^{||} \;=\; \frac{1}{\mu_2} B_2^{||}$$

$$\text{But} \quad \tan\theta_1 \;=\; \frac{B_1^{||}}{B_1^{\perp}} \quad \text{and} \quad \tan\theta_2 \;=\; \frac{B_2^{||}}{B_2^{\perp}}$$

$$=> \quad \frac{\tan\theta_2}{\tan\theta_1} \;=\; \frac{B_1^{\perp}}{B_1^{||}} \frac{B_2^{||}}{B_2^{\perp}} \;=\; \frac{B_2^{||}}{B_1^{||}} \;=\; \frac{\mu_2}{\mu_1}$$