

Physics 322 Solution to Homework Set #4 Spring 2008

1. Problem 6.1 in your textbook. Let $\vec{m}_1 = \pi a^2 I \hat{z}$ be the magnetic dipole moment of the circular loop and $\vec{m}_2 = b^2 I \hat{x}$ be the magnetic dipole moment of the square loop. Let \vec{B}_1 be the magnetic field produced by the circular loop. Then:

$$\begin{aligned} \vec{N}_2 &= \vec{m}_2 \times \vec{B}_1; & \vec{B}_1 &= \frac{\mu_0}{4\pi r^3} [3(\vec{m}_1 \cdot \hat{r})\hat{r} - \vec{m}_1] \quad \text{where } \hat{r} = \hat{x} \Rightarrow \vec{B}_1 = -\frac{\mu_0 \pi a^2 I}{4\pi r^3} \hat{z} \\ & \Rightarrow \vec{N}_2 &= -\frac{b^2 I \mu_0 \pi a^2 I}{4\pi r^3} (\hat{x} \times \hat{z}) = \frac{\mu_0 (abI)^2}{4r^3} \hat{y} \end{aligned}$$

The final orientation of the square loop will have its dipole moment pointing down, along $-\hat{z}$.

2. Problem 6.3 in your textbook.

(a) Let the two dipoles lie along the \hat{z} axis. From Eq. 6.2, $F = 2\pi IRB \cos \psi$ where \vec{B} is the field produced by the solenoid (the field produced by \vec{m}_1 for this problem), R is the radius of the circular loop ($\vec{m}_2 = \pi R^2 I \hat{z}$ for this problem), and ψ is the angle between \vec{B} and the radial axis at the location of the circular loop. Let's call the radial direction \hat{y} . Then $B \cos \psi = \vec{B} \cdot \hat{y}$.

To use Eq. 6.2, we consider \vec{m}_2 to be a circular loop of radius R . Let \vec{r} be the vector from \vec{m}_1 to a point on the circular loop of \vec{m}_2 . Then,

$$\vec{B} = \frac{\mu_0}{4\pi r^3} [3(\vec{m}_1 \cdot \hat{r})\hat{r} - \vec{m}_1] \quad \text{and} \quad \vec{B} \cdot \hat{y} = B \cos \psi = \frac{\mu_0}{4\pi r^3} [3(\vec{m}_1 \cdot \hat{r})(\hat{r} \cdot \hat{y}) - (\vec{m}_1 \cdot \hat{y})]$$

But $\vec{m}_1 \cdot \hat{y} = 0$ and $\vec{m}_1 \cdot \hat{r} = m_1 \cos \theta$, $\hat{r} \cdot \hat{y} = \sin \theta$, where θ is the usual polar angle of \hat{r} . Therefore,

$$B \cos \psi = \frac{\mu_0}{4\pi r^3} [3m_1 \cos \theta \sin \theta] \Rightarrow F = 2\pi IRB \cos \psi = \frac{\mu_0 IR}{2r^3} [3m_1 \cos \theta \sin \theta]$$

Finally, $\sin \theta = R/r$ and $\cos \theta = \sqrt{r^2 - R^2}/r$. Using $m_2 = \pi IR^2$ as $R \rightarrow 0$ for a point dipole,

$$F = \frac{3\mu_0 m_1 IR^2}{2} \frac{\sqrt{r^2 - R^2}}{r^5} = \frac{3\mu_0 m_1 m_2}{2\pi} \frac{\sqrt{r^2 - R^2}}{r^5} \rightarrow \frac{3\mu_0 m_1 m_2}{2\pi r^4} \text{ as } R \rightarrow 0$$

where the force is attractive.

(b)

$$\vec{F} = \vec{\nabla}(\vec{m}_2 \cdot \vec{B}) = (\vec{m}_2 \cdot \vec{\nabla})\vec{B} = \left(m_2 \frac{d}{dz}\right) \left[\frac{\mu_0}{4\pi z^3} [3(\vec{m}_1 \cdot \hat{z})\hat{z} - \vec{m}_1]\right]$$

But $3(\vec{m}_1 \cdot \hat{z})\hat{z} - \vec{m}_1 = 2\vec{m}_1 \Rightarrow$

$$\vec{F} = \frac{\mu_0 m_1 m_2 \hat{z}}{2\pi} \frac{d}{dz} \left(\frac{1}{r^3}\right) = -\frac{3\mu_0 m_1 m_2}{2\pi z^4} \hat{z} = -\frac{3\mu_0 m_1 m_2}{2\pi r^4} \hat{z} \quad \text{because } z = r$$

3. Problem 6.8 in your textbook.

$$\vec{K}_b = \vec{M} \times \hat{n} = ks^2(\hat{\phi} \times \hat{s}) = -kR^2 \hat{z} \quad \text{and} \quad \vec{J}_b = \vec{\nabla} \times \vec{M} = \frac{1}{s} \frac{\partial}{\partial s} (sk s^2) \hat{z} = \frac{1}{s} (3ks^2) \hat{z} = 3ks \hat{z}$$

So the bound current flows up the cylinder and returns down the surface. (The total current is zero as it should be because $\int J_b da = \int_0^R (3ks)(2\pi s ds) = 2\pi kR^3$ while $\int K_b dl = (-kR^2)(2\pi R) = -2\pi kR^3$). Because these currents have cylindrical symmetry, we can find \vec{B} by using Ampere's law. Inside the cylinder,

$$B \cdot 2\pi s = \mu_0 I_{enc} = \mu_0 \int_0^s J_b da = 3\mu_0 k \int_0^s s'(2\pi s' ds') = 2\pi k \mu_0 s^3 \Rightarrow \vec{B} = \mu_0 k s^2 \hat{\phi} = \mu_0 \vec{M}$$

Outside the cylinder, $I_{enc} = 0 \Rightarrow B = 0$.

4. Problem 6.10 in your textbook.

The field inside a ring with no gap is $\vec{B} = \mu_0(\vec{H} + \vec{M}) = \mu_0\vec{M}$ because there are no free currents. By superposition, the field in the gap, $\vec{B}_{gap} = \vec{B} - \vec{B}_{rem}$ where \vec{B}_{rem} is the field of the uniformly magnetized piece that was removed from the gap. Because the removed piece has uniform magnetization, its field is the field of a square current loop with current $I = K_b w = Mw$. From problem 5.8, the field at the center of a square current loop is $B_{sq} = \sqrt{2}\mu_0 I / (\pi R)$ where $R = a/2$. Therefore:

$$B_{sq} = \frac{\sqrt{2}\mu_0 Mw}{\pi(a/2)} = \frac{2\sqrt{2}\mu_0 Mw}{\pi a} \Rightarrow \vec{B}_{gap} = \mu_0\vec{M} \left(1 - \frac{2\sqrt{2}w}{\pi a} \right)$$

5. Problem 6.12 in your textbook.

(a) Like problem 3 above, with $\vec{M} = ks\hat{z}$,

$$\vec{K}_b = \vec{M} \times \hat{n} = ks(\hat{z} \times \hat{s}) = kR\hat{\phi} \quad \text{and} \quad \vec{J}_b = \vec{\nabla} \times \vec{M} = -\frac{\partial}{\partial s}(ks)\hat{\phi} = -k\hat{\phi}$$

again with the total bound current equal and opposite to the total surface current. This is the superposition of two infinite solenoids, giving us $B = 0$ outside the cylinder and an axial field inside. For the field inside the cylinder, we take an amperian loop with one side (length L outside of the cylinder and the other inside the cylinder at a distance s from the axis:

$$\begin{aligned} \int_C \vec{B} \cdot d\vec{l} = B_{in}L = \mu_0 I_{enc} = \mu_0 \left[\int J_b da + K_b L \right] &= \mu_0 \left[L \int_s^R (-k) ds' + K_b L \right] = \mu_0 \left[-kL(R-s) + kRL \right] \\ \Rightarrow B_{in}L = \mu_0 kLs &\Rightarrow \vec{B}_{in} = \mu_0 ks\hat{z} = \mu_0\vec{M} \end{aligned}$$

(b) $\int \vec{H} \cdot d\vec{l} = I_{free,enc} = 0 \Rightarrow \vec{H} = 0$ because there are no free currents. (This works for an infinite cylinder but not a finite one where $\vec{\nabla} \cdot \vec{M} \neq 0$ at the ends.) $\vec{B} = \mu_0[\vec{H} + \vec{M}] = \mu_0\vec{M}$ inside of the cylinder and $B = 0$ outside where M vanishes.