Physics 322 Solution to Homework Set #2 Spring 2008

1. Problem 5.9 in your textbook. We use the Biot-Savart law:

$$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\vec{l} \times \hat{r}}{r^2}$$

(a) The straight sections generate no field because $d\vec{l}$ is parallel to $\vec{r}$, while over the quarter circles $d\vec{l}$ is perpendicular to $\vec{r}$ giving us $d\vec{l} \times \hat{r} = dl$ pointing out of the page for the inner circle and into the page for the outer ($\vec{r}$ points from $d\vec{l}$ to point P). Therefore,

$$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\vec{l} \times \hat{r}}{r^2} = \frac{\mu_0 I}{4\pi} \Big[\frac{2\pi a}{4} \frac{1}{a^2} - \frac{2\pi b}{4} \frac{1}{b^2}\Big] = \frac{\mu_0 I}{8} \Big(\frac{1}{a} - \frac{1}{b}\Big)$$

pointing out of the page.

(b) Again, integrating the Biot-Savart law, the two half lines sum up to the field from an infinite line of current, $B = \mu_0 I/(2\pi R)$, while the half circle contributes one-half of the field from a circular loop, $B = \mu_0 I/(4R)$. Therefore,

$$\vec{B} = \frac{\mu_0 I}{2\pi R} + \frac{\mu_0 I}{4R} = \frac{\mu_0 I}{4R}\Big(1 + \frac{2}{\pi}\Big)\hat{z}$$

where $\hat{z}$ points into the page.

2. Problem 5.10 in your textbook. We use:

$$\vec{F}_{mag} = I \int (d\vec{l} \times \vec{B})$$

(a) The field from the long wire points out of the page (call this $\hat{z}$), and let the current in the long wire flow in the $\hat{x}$ direction. The force it produces on the sections of the square loop that are perpendicular to the long wire cancel (because I is opposite on the 2 sides). The force on the bottom section of the square (closest to the long wire) is $\vec{I}a \times \vec{B} = -Ia\hat{x} \times \mu_0 I/(2\pi s)\hat{z} = \mu_0 I^2 a/(2\pi s)\hat{y}$. The force on the top section ofthe square is $\vec{I}a \times \vec{B} = Ia\hat{x} \times \mu_0 I/(2\pi(s+a))\hat{z} = -\mu_0 I^2 a/(2\pi(s+a))\hat{y}$. The total force is then:

$$\vec{F}_{mag} = \frac{\mu_0 I^2 a}{2\pi}\Big[\frac{1}{s} - \frac{1}{s+a}\Big]\hat{y} = \frac{\mu_0 I^2 a^2}{2\pi s(s+a)}\hat{y}$$

where $\hat{y}$ points away from the long wire (up in the diagram).

(b) The force on the bottom section is the same as above: $\mu_0 I^2 a/(2\pi s)\hat{y}$. For the angled section on the left,

$$d\vec{F} = I(d\vec{l} \times \vec{B}) = I(dx\ \hat{x} + dy\ \hat{y} + dz\ \hat{z}) \times \frac{\mu_0 I \hat{z}}{2\pi y} = \frac{\mu_0 I^2}{2\pi y}(-dx\ \hat{y} + dy\ \hat{x})$$

while for the angled section on the right, dx is in the same direction while dy changes sign, making the $\hat{x}$ component of the force cancel and the $\hat{y}$ components add. The total force from the two angled sections is then:

$$\vec{F} = -\frac{2\mu_0 I^2 \hat{y}}{2\pi} \int \frac{dx}{y} \quad \text{but} \quad \frac{dy}{dx} = \tan 60° = \sqrt{3} \quad => \quad \vec{F} = -\frac{2\mu_0 I^2 \hat{y}}{2\pi} \int_s^{s+a\sqrt{3}/2} \frac{dy}{\sqrt{3}\ y}$$

(the height of the equilateral triangle is $a\sqrt{3}/2$). Adding all 3 sections of the loop,

$$\vec{F} = \frac{\mu_0 I^2 \hat{y}}{2\pi}\Big[\frac{a}{s} - \frac{2}{\sqrt{3}} \ln\Big(\frac{s + a\sqrt{3}/2}{s}\Big)\Big] = \frac{\mu_0 I^2 \hat{y}}{2\pi}\Big[\frac{a}{s} - \frac{2}{\sqrt{3}} \ln\Big(1 + \frac{\sqrt{3}a}{2s}\Big)\Big]$$

3. Problem 5.13 in your textbook. We use Ampere's law for a circular loop centered around the axis of the wire, knowing that the field from a straight wire runs circumferentially around the wire:

(a) $\int_C \vec{B} \cdot d\vec{l} = B2\pi s = \mu_0 I_{enc} \Rightarrow B = 0$ for $s < a$ and $\vec{B} = \mu_0 I/(2\pi s)\hat{\phi}$ for $s > a$.

(b) $J = ks$ and $I = \int J\, da = \int_0^a ks(2\pi s)ds = 2\pi k a^2/3 \Rightarrow k = 3I/(2\pi a^3)$. For $s < a$,

$$I_{enc} = \int_0^s J\, da = \int_0^s ks'(2\pi s')ds' = \frac{2\pi k s^3}{3} = I\frac{s^3}{a^3}$$

while for $s > a$, $I_{enc} = I$. Using Ampere's law,

$$\vec{B} = \frac{\mu_0 I s^2}{2\pi a^3}\hat{\phi} \quad \text{for} \quad s < a \qquad \text{and} \qquad \vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi} \quad \text{for} \quad s > a$$

4. Problem 5.14 in your textbook. By the right hand rule, the field points along $-\hat{y}$ for $z > 0$ and along $+\hat{y}$ for $z < 0$. For $z = 0$, $B = 0$ because the fields from the currents above and below $z = 0$ cancel by symmetry. Take an Amperian current loop where one side of length L runs parallel to $\hat{y}$ with $z = 0$ and the other side, parallel to $\hat{y}$, is a height z above the $z = 0$ segment. For $z < a$, the current enclosed in the loop is $I_{enc} = \int \vec{J} \cdot d\vec{a} = JLz$ while for $z > a$, $I_{enc} = JLa$.

We now use Ampere's law: $\int_C \vec{B} \cdot d\vec{l} = BL$ because the segment along $z = 0$ contributes 0 and the segments parallel to $\hat{z}$ contribute 0 ($d\vec{l}$ is perpendicular to $\vec{B}$). Therefore:

$$\text{For} \quad -a < z < a \quad \vec{B} = -\mu_0 J z \hat{y} \qquad \text{for} \quad z > a \quad \vec{B} = -\mu_0 J a \hat{y} \qquad \text{and for} \quad z < -a \quad \vec{B} = +\mu_0 J a \hat{y}$$

5. Problem 5.15 in your textbook. The field inside a solenoid is $\mu_0 n I$ and the field outside is 0. We use superposition, noting that the field of the outer solenoid points to the right ($+\hat{z}$) while the inner solenoid's field points to the left ($-\hat{z}$).

$$\text{For} \quad r < a \quad \vec{B} = \mu_0 I(n_2 - n_1)\hat{z}, \qquad \text{for} \quad a < r < b \quad \vec{B} = \mu_0 I n_2 \hat{z} \qquad \text{and for} \quad r > b \quad \vec{B} = 0$$

6. Problem 5.16 in your textbook. Let's use the coordinate system of example 5.8 in the text, where $\hat{z}$ is perpendicular to the plates and the surface current density is in the $\hat{x}$ direction. Then the top plate makes a surface current density $\vec{K}_t = \sigma v \hat{x}$ and the bottom plate a surface current density $\vec{K}_b = -\sigma v \hat{x} = -\vec{K}_t$.

(a) From example 5.8 in the text, the top plate produces a field above the top plate $\vec{B}_{t+} = -\mu_0 K_t/2\ \hat{y}$ and a field below the top of $\vec{B}_{t-} = \mu_0 K_t/2\ \hat{y}$. Similarly, the bottom plate produces a field above the bottom plate of $\vec{B}_{b+} = -\mu_0 K_b/2\ \hat{y} = \mu_0 K_t/2\ \hat{y}$ and a field below the bottom plate of $\vec{B}_{b-} = \mu_0 K_t/2\ \hat{y} = -\mu_0 K_t/2\ \hat{y}$.

By superposition we find that above both or below both plates, $\vec{B} = \vec{B}_t + \vec{B}_b = \propto \vec{K}_t + \vec{K}_b = 0$, while between the plates, $\vec{B} = \vec{B}_{t-} + \vec{B}_{b+} = \mu_0(K_t/2\ \hat{y} + K_t/2\ \hat{y}) = \mu_0 \sigma v\ \hat{y}$ (into the page).

(b) the Lorentz force law says $\vec{F}_m = \int(\vec{K} \times \vec{B})\ da$, so the force per unit area is $\vec{f}_m = \vec{K} \times \vec{B}$. Here, $\vec{K} = \vec{K}_t = \sigma v\ \hat{x}$ and $\vec{B} = \vec{B}_{b+} = \mu_0 \sigma v/2\ \hat{y} \Rightarrow \vec{f}_m = \mu_0 \sigma^2 v^2/2\ (\hat{x} \times \hat{y}) = \mu_0 \sigma^2 v^2/2\ \hat{z}$ (up in the diagram).

(c) The electric field of the lower plate is $\sigma/(2\epsilon_0)$ and the electrical force per unit area on the top plate is $f_e = \sigma^2/(2\epsilon_0)$. (directed down). This balances the magnetic force if $\mu_0 v^2 = 1/\epsilon_0$ or $v = 1/\sqrt{\epsilon_0 \mu_0} = c$, the speed of light.