Physics 322 Solution to Homework Set #1 Spring 2008

Due in class 4/11/08

1. Because $\vec{v} \times \vec{B}$ points up as does $\vec{F}$, the charge must be positive.

To find the momentum, we know that the trajectory of the particle in the field region is a circle of radius R. We first find R: The center of the circular orbit is a distance R above the point where the charge enters the field region – draw this initial vector, $\vec{R}_i$. The vector, $\vec{R}_f$, from the orbit center to the point d where the charge leaves the field region also has length R, and let θ be the angle between $\vec{R}_i$ and $\vec{R}_f$.

By drawing a line perpendicular to $\vec{R}_i$ that passes through point d, we see that $\sin\theta = a/R$ and $\cos\theta = (R-d)/R$.

$$\sin^2\theta + \cos^2\theta \;=\; 1 \;=\; \frac{a^2}{R^2} + \frac{(R-d)^2}{R^2} \quad => \quad a^2 + (R-d)^2 \;=\; R^2 \quad => \quad R \;=\; \frac{a^2+d^2}{2d}$$

Using $p = qBR$, we find that $p = qB(a^2+d^2)/(2d)$.

2. Problem 5.2 in your textbook. The general solutions are:

$$y(t) \;=\; C_1\cos(\omega t) + C_2\sin(\omega t) + \frac{E}{B}t + C_3 \qquad z(t) \;=\; C_2\cos(\omega t) - C_1\sin(\omega t) + C_4$$

(a) $y(0) = z(0) = \dot{z}(0) = 0$ and $\dot{y}(0) = E/B$.

$$=> \quad C_1 + C_3 \;=\; C_2 + C_4 \;=\; -\omega C_1 \;=\; 0 \quad \text{and} \quad \omega C_2 + \frac{E}{B} \;=\; \frac{E}{B}$$

Therefore, $C_1 = 0$ which makes $C_3 = 0$ and $C_2 = 0$ which makes $C_4 = 0$, leaving us with:

$$y(t) \;=\; \frac{E}{B}t \quad \text{and} \quad z(t) \;=\; 0$$

This makes sense: at this special velocity, $\vec{v} = (E/B)\hat{y}$, the magnetic force has a magnitude of E and points in the $-\hat{z}$ direction, just cancelling the electric force. The net force on the charge is therefore 0, so the charge moves along the $\hat{y}$ direction at constant speed. (Sending a beam of charged particles through perpendicular electric and magnetic fields allows only those particles with velocity = E/B to pass through undeflected, making a velocity filter for the beam.)

(b) We have the same initial conditions as above, except that:

$$\dot{y}(0) = \omega C_2 + E/B = E/(2B) \quad => \quad C_2 = -E/(2\omega B)$$

. Like above, $\dot{z}(0) = 0 => C_1 = 0$, which gives us $C_3 = 0$, and $C_4 = -C_2 = E/(2\omega B)$. Therefore:

$$y(t) \;=\; -\frac{E}{2\omega B}\sin(\omega t) + \frac{E}{B}t \;=\; \frac{E}{2\omega t}\left[2\omega t - \sin(\omega t)\right] \quad \text{and} \quad z(t) \;=\; \frac{E}{2\omega B}\left[1 - \cos(\omega t)\right]$$

Letting $\beta = E/(2\omega B)$, we have:

$$(y(t) - 2\beta\omega t) \;=\; -\beta\sin(\omega t) \quad \text{and} \quad (z(t) - \beta) \;=\; -\beta\cos(\omega t)$$

$$=> \quad \left[y(t) - 2\beta\omega t\right]^2 + \left[z(t) - \beta\right]^2 \;=\; \beta^2$$

This is the equation of a circle whose center, (y_0, z_0) moves to the right at constant speed: $(y_0, z_0) = (2\beta\omega t, \beta)$.

(c)

$$\dot{z}(0) = \frac{E}{B} \quad => \quad -\omega C_1 = \frac{E}{B} \quad \text{and} \quad y(0) = 0 \quad => \quad C_3 = -C_1 = -\frac{E}{B}$$

$$\dot{y}(0) = \frac{E}{B} = \omega C_2 + \frac{E}{B} \quad => \quad C_2 = 0 = -C_4 \quad (\text{from } z(0) = 0)$$

$$\text{Therefore} \quad y(t) = \frac{E}{\omega B}\big[1 + \omega t - \cos(\omega t)\big] \quad \text{and} \quad z(t) = \frac{E}{\omega B}\sin(\omega t)$$

Like above, we let $\gamma = E/(\omega B)$. Then,

$$\big[y - \gamma(1+\omega t)\big] = -\gamma\cos(\omega t) \quad \text{and} \quad z = \gamma\sin(\omega t) \quad => \quad \big[y - \gamma(1+\omega t)\big]^2 + z^2 = \gamma^2$$

This is the equation of a circle whose center, (y_0, z_0) moves to the right at constant speed: $(y_0, z_0) = (\gamma(1+\omega t), 0)$.

3. Problem 5.6 in your textbook.
(a) The surface current density $\vec{K} = \sigma\vec{v}$ (Eqn. 5.23 in the text). For the phonograph record, $v = \omega r$ giving us $K = \sigma\omega r$.

(b) The volume current density $\vec{J} = \rho\vec{v}$. For the uniformly charged sphere, $\rho = Q/\text{Vol} = 3Q/(4\pi R^3)$. The speed of any charge element is the angular velocity times the distance from the rotation ($\hat{z}$) axis: $v = \omega r\sin\theta$. Therefore, $J = \rho\omega r\sin\theta = 3Q\omega r\sin\theta/(4\pi R^3)$. The direction of $\vec{J}$ is the direction of $\vec{v}$ which is the $\hat{\phi}$ direction in spherical coordinates.

4. Problem 5.8 in your textbook. (a) By symmetry (and the right-hand-rule), the field at the center of the square loop points up, perpendicular to the loop. We need to integrate the Biot-Savart law around the square, but all four sides are equivalent, so we only need to integrate along one side (and multiply the result by 4). Your text does this integration (Example 5.5) for a straight section of wire, so we can use the results from the text (Eqn. 5.35):

$$B = \frac{\mu_0 I}{4\pi s}(\sin\theta_2 - \sin\theta_1)$$

where s is the perpendicular distance from the field point to the line segment ($s = R$ for our problem) and θ_1 and θ_2 are initial and final angles of the ends of the current segment relative to the field point (see Figure 5.18). For our problem, $\theta_1 = -45°$ and $\theta_2 = 45°$

$$=> \quad B = 4 \times \frac{\mu_0 I}{4\pi R}(2/\sqrt{2}) = \frac{\sqrt{2}\mu_0 I}{\pi R}$$

(b) We use the same equation as above with $s = R$, but now, $\theta_1 = -\pi/n$ and $\theta_2 = \pi/n$.

$$B = n \times \frac{\mu_0 I}{4\pi R}[2\sin(\pi/n)] = \frac{n\mu_0 I}{2\pi R}\sin(\pi/n)$$

(c) For small θ, $\sin\theta \approx \theta$

$$=> \quad \text{as} \quad n \to \infty, \quad B \to \frac{n\mu_0 I}{2\pi R}\frac{\pi}{n} = \frac{\mu_0 I}{2R}$$

which is the result for a circular loop of radius R.

5. Problem 5.11 in your textbook. Let the axis of the solenoid be $\hat{z}$, and consider a ring of the solenoid of width dz a distance z from the observation point, P. We can use Eqn. 5.38 for the field, dB_z, from the ring where I in Eqn. 5.38 becomes $nI\ dz$ for our ring (giving us a current of nI per unit length as specified).

$$dB_z(P) \;=\; \frac{\mu_0 nIdz}{2}\,\frac{a^2}{(a^2+z^2)^{3/2}} \qquad => \qquad B_z(P) \;=\; \frac{\mu_0 nI}{2}\int_{z_1}^{z_2}\frac{a^2\ dz}{(a^2+z^2)^{3/2}}$$

Using the angles θ defined in Fig. 5.25, we have $a/z = \tan\theta$, or better, $z/a = \cot\theta$. Therefore:

$$dz \;=\; \frac{a}{\sin^2\theta}d\theta \quad \text{and} \quad \frac{1}{(a^2+z^2)^{3/2}} \;=\; \frac{1}{a^3(1+\cos^2\theta/\sin^2\theta)^{3/2}} \;=\; \frac{\sin^3\theta}{a^3}$$

$$=> \quad B_z(P) \;=\; \frac{\mu_0 nI}{2}\int_{\theta_2}^{\theta_1}\frac{a^2\sin^3\theta}{a^3\sin^2\theta}(a\ d\theta) \;=\; \frac{\mu_0 nI}{2}\int_{\theta_1}^{\theta_2}\sin\theta\ d\theta \;=\; \frac{\mu_0 nI}{2}(\cos\theta_2-\cos\theta_1)$$

For an infinite solenoid, the point P lies within the solenoid and $\theta_2 = 0$ and $\theta_1 = \pi$, giving us $B_z = \mu_0 nI$.