Physics 321 Solution to Final Exam Winter 2008

This is an open book (textbook only), open notes exam. Show all of your work to receive full credit. The exam is worth 100 pts.

Part I. (18 pts) Shielding A hollow metal (conducting) box has zero net charge on it. A point charge, $+Q$, is placed inside of the conducting box, as shown below. **You must explain your answers to receive full credit**.

1. (6 pts) Circle the **true** statements below. Explain.

a) The electric field is zero inside of the box.

False, charge Q produces an electric field.

b) The box is an equipotential surface (V_{box} = constant).

True, E must be zero within the conductor, requiring a constant potential on its surface.

c) $V_{box} = V_\infty$

False, Gauss's law tells us that $E \neq 0$ outside of the box, so $V_{box} \neq V_\infty$.

d) There is an induced charge on the inner surface of the box.

True, Q induces an image charge $-Q$ on the inner surface of the box (to make $E = 0$ inside of the box), leaving $+Q$ on the outer surface.

2. (4 pts) A second charge, $+q$, is placed a outside of the box, a distance, s, away with $s >> b$. Does the box (with $+Q$ inside it) exert a force on the second charge, $+q$?. If so, is the force attractive or repulsive? Explain.

Yes, the box exerts a repulsive force on the second charge because the outer surface of the box carries a charge $+Q$ to balance the induced charge $-Q$ on the inner surface of the box. Far from the box, the field of the box approaches the field from a point charge $+Q$, leading to a repulsive force on $+q$.

3. (4 pts) Does the second charge, $+q$, exert a force on the charge $+Q$ in the box? Explain.

No, the second charge exerts a force on the box, but not on $+Q$ inside of the box. The condition that $E = 0$ in the walls of the box shields the inside of the box from fields outside of the box.

4. (4 pts) The metal box (with $+Q$ inside of it) is now grounded (*i.e.* its electric potential is set to zero). Does the grounded box (with $+Q$ inside it) exert a force on the second charge, $+q$?. If so, is the force attractive or repulsive? Explain.

Yes, but now the force is attractive. Grounding the box removed the charge $+Q$ that was on its outer surface. The second charge induces an opposite sign charge on the box surface, leading to an attractive force (like a charge above a conducting plane).

Part II. (15 pts) Gauss's law and Polarizability Assume that the electron cloud in a neutral atom of atomic number Z (nucleus charge of $+Ze$) has a charge density, $\rho(r)$, given by:

$$\rho(r) = \rho_0 \; e^{-r/a} \quad \text{where} \quad \rho_0 \;=\; \frac{-Ze}{8\pi a^3} \quad => \quad \int \rho(r)\; d\tau \;=\; -Ze$$

1. (5 pts) Use the fact that for $r << a$, $e^{-r/a} \approx 1$ to show that the electric field of the electron cloud near the nucleus is given by $E(r) = \rho_0 r/(3\epsilon_0)$.

By symmetry, the electron cloud $\vec{E}$ points radially inward. We use Gauss's law on a small sphere of radius R centered on the origin to find :

$$\int_S \vec{E}\cdot d\vec{a} \;=\; 4\pi R^2 E(R) \;=\; \frac{Q_{enc}}{\epsilon_0} \;=\; \frac{1}{\epsilon_0}\int_V \rho(r) d\tau \;=\; \frac{4\pi}{\epsilon_0}\int_0^R \rho_0 \; e^{-r/a}\; r^2\; dr \;\approx\; \frac{4\pi}{\epsilon_0}\int_0^R \rho_0 \; r^2 \; dr$$

$$=> \quad 4\pi R^2 E(R) \;=\; \frac{4\pi}{\epsilon_0}\rho_0 \frac{r^3}{3}\Big|_0^R \;=\; \frac{4\pi\rho_0 R^3}{3\epsilon_0} \quad => \quad E(R) \;=\; \frac{\rho_0 R}{3\epsilon_0}$$

2. (4 pts) Use the result from part **1** to estimate the atomic polarizability, α .

$\vec{p} = \alpha \vec{E}_{ext}$ The external field, $\vec{E}_{ext}$ exerts a force on the nucleus, causing the nucleus to move a distance d until $\vec{E}_{ext}$ is cancelled by the field from the electron cloud:

$$E_{ext} \;=\; -E(d) \;=\; -\frac{\rho_0 d}{3\epsilon_0} \quad => \quad d \;=\; -\frac{3\epsilon_0 E_{ext}}{\rho_0}$$

But the dipole moment of the atom is $p = qd = Zed$:

$$=> \quad p \;=\; Zed \;=\; -\frac{3Ze\;\epsilon_0 E_{ext}}{\rho_0} \quad => \quad \alpha \;=\; -\frac{3Ze\;\epsilon_0}{\rho_0} \;=\; 24\pi\epsilon_0 a^3$$

3. (6 pts) Calculate the electric potential at a distance a from the nucleus, $V(a)$. You may find the following integral useful: $\int x^2 e^{-x/a} dx = -a e^{-x/a}[x^2 + 2ax + 2a^2]$. (Assume $V(\infty) = 0$).

Inside of a spherical charge distribution, the potential at R is that of a point charge at the center of the sphere whose charge equals the charge within the volume of radius R:

$$V(R) \;=\; \frac{Q_{enc}}{4\pi\epsilon_0 R} \;=\; \frac{4\pi\rho_0}{4\pi\epsilon_0 R}\int_0^R e^{-r/a}\; r^2 dr \;=\; \frac{\rho_0}{\epsilon_0 R}\;\big[-a e^{-r/a}(r^2 + 2ar + 2a^2)\big]_0^R$$

$$=> \quad V(a) \;=\; \frac{\rho_0}{\epsilon_0 R}\;\big[-a e^{-1}(5a^2) + 2a^3\big] = \frac{\rho_0 a^3}{\epsilon_0 R}\big[2 - 5e^{-1}\big]$$

That is the potential from the electron cloud. The potential from the nucleus at a distance, a, is $V_{Nuc} = Ze/(4\pi\epsilon_0 a)$. The total potential is the sum of the two pieces.

Part III. (18 pts) Dielectrics A conducting metal sphere of radius a with a charge $+Q$ on it is surrounded by a spherical dielectric shell with outer radius b (inner radius a). The dielectric shell has no net charge and has a relative dielectric constant ϵ_r.

1. (3 pts) Find the displacement field, $\vec{D}$, everywhere.

For $r < a$, $\vec{E} = \vec{D} = 0$

For $r > a$, Gauss's law, $\int_S \vec{D} \cdot d\vec{a} = Q_{enc}$ => $\vec{D} = Q/(4\pi r^2)\hat{r}$

2. (3 pts) Find the electric field, $\vec{E}$, everywhere.

For $r < a$, $\vec{E} = \vec{D} = 0$

For $a < r < b$, $\vec{E} = \vec{D}/(\epsilon_r \epsilon_0) = Q/(4\pi\epsilon_r\epsilon_0 r^2)\hat{r}$

For $r > b$, $\vec{E} = \vec{D}/(\epsilon_0) = Q/(4\pi\epsilon_0 r^2)\hat{r}$

3. (3 pts) Find the polarization density, $\vec{P}$, within the dielectric.

$$\vec{P} \;=\; \epsilon_0 \chi_e \vec{E} \;=\; \epsilon_0(\epsilon_r - 1)\vec{E} \;=\; \frac{(\epsilon_r - 1)Q}{4\pi\epsilon_r r^2}\,\hat{r}$$

4. (3 pts) Find the surface charge, σ_b, on the inner and outer surfaces of the dielectric.

$$\sigma_b(b) \;=\; \vec{P}\cdot\hat{n} \;=\; \vec{P}\cdot\hat{r} \;=\; \frac{(\epsilon_r - 1)Q}{4\pi\epsilon_r b^2}$$

$$\sigma_b(a) \;=\; \vec{P}\cdot\hat{n} \;=\; -\vec{P}\cdot\hat{r} \;=\; -\frac{(\epsilon_r - 1)Q}{4\pi\epsilon_r a^2}$$

5. (3 pts) Find the bound charge, ρ_b within the dielectric.

$$\rho_b \;=\; -\vec{\nabla}\cdot\vec{P} \;\propto\; \vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) \;=\; 0 \quad \text{because} \quad r \neq 0$$

6. (3 pts) Find the electric potential, $V(b)$, at the outer surface of the dielectric.

$$V(b) \;=\; \frac{Q}{4\pi\epsilon_0 b}$$

because for $r > b$, the electric field is the same as that of a point charge Q at the origin, and $V(b)$ is the work required to move a unit charge from infinity to the surface of the sphere.

Part IV. (15 pts) Separation of Variables
A rectangular pipe, running parallel to the z axis (from $-\infty$ to $+\infty$) has three grounded metal sides, at $y = 0$, $y = a$, and $x = 0$. The fourth side, at $x = b$, is maintained at a specified potential $V_0(y)$.

1. (8 pts) Develop a general formula for the potential within the pipe, valid for all $V_0(y)$.

The general solution to the Laplace equation in two dimensions is:

$$V(x,y) = \big(A\cosh kx + B\sinh kx\big)\big(C\cos ky + D\sin ky\big)$$

where we chose the sinusoidal solution for the $\hat{y}$ direction because the potential vanishes along 2 surfaces parallel to $\hat{y}$.

$$V(x,0) = 0 \quad => \quad C = 0 \qquad \text{and} \qquad V(0,y) = 0 \quad => \quad A = 0$$

$$V(x,a) = 0 \quad => \quad ka = n\pi$$

$$\text{Therefore} \quad V(x,y) = \sum_{n=1}^{\infty} B_n \sinh(n\pi x/a)\ \sin(n\pi y/a)$$

To make $V = V_0(y)$ at $x = \pm b$, we need:

$$V(b,y) = V_0(y) = \sum_{n=1}^{\infty} B_n \sinh(n\pi b/a)\ \sin(n\pi y/a)$$

Multiplying both sides by $\sin(m\pi y/a)$ and integating over y gives us:

$$B_n = \frac{2}{a\sinh(n\pi b/a)} \int_0^a V_0(y)\sin(n\pi y/a)\ dy$$

2. (7 pts) Find the potential within the pipe explicitly for the case $V_0(y) = V_0 = \text{constant}$.

$$B_n = \frac{2}{a\sinh(n\pi b/a)} V_0 \int_0^a \sin(n\pi y/a)\ dy = \frac{2V_0}{a\sinh(n\pi b/a)} \cdot \frac{2a}{n\pi} \quad \text{for } n \text{ odd,} \quad 0 \text{ for } n \text{ even}$$

$$V(x,y) = \frac{4V_0}{\pi} \sum_{n=1,3,5,\ldots} \frac{\sinh(n\pi x/a)\sin(n\pi y/a)}{n\ \sinh(n\pi b/a)}$$

Part V. (16 pts) Capacitance and Dielectrics A cable of length L consists of two coaxial cylindrical conductors, the inner having radius a and the outer having radius b with $b << L$. One end of the cable is connected to a battery which maintains a potential, V_0, on the inner conductor while the outer conductor is grounded ($V(b) = 0$). The other end of the cable is open, and initially the space between the cylinders is filled with air (whose dielectric properties may be neglected).

1. (4 pts) Find the capacitance of the cable (neglecting end effects) using Gauss's law or otherwise.

Taking a Gaussian cylinder of length L and radius r enclosing the inner conductor:

$$\int_S \vec{E} \cdot d\vec{a} \;=\; \frac{Q_{enc}}{\epsilon_0} \;=\; 2\pi r L E(r) \quad => \quad E(r) \;=\; \frac{Q_{enc}}{2\pi r \epsilon_0 L}$$

$$V_0 \;=\; \int_b^a -E(r)dr \;=\; \int_a^b \frac{Q_{enc}}{2\pi r \epsilon_0 L} dr \;=\; \frac{Q_{enc}}{2\pi \epsilon_0 L} \ln(b/a) \quad => \quad C_0 \;=\; \frac{Q_{enc}}{V_0} \;=\; \frac{2\pi \epsilon_0 L}{\ln(b/a)}$$

2. (4 pts) The cable is immersed in oil with relative dielectric constant, ϵ_r, and the cable fills with oil. Find the electrical work done by the battery during this process. (Again, neglect end effects and gravity.)

Immersed in oil, the capacitance of the cable, $C' = \epsilon_r C_0$. Because the voltage on the cable is still V_0, an amount of charge $\Delta Q = Q' - Q_0 = C'V_0 - C_0 V_0 = (C' - C_0)V_0$ had to flow onto the inner conductor. The work done by the battery is then:

$$W \;=\; \Delta U \;=\; \Delta Q \; V_0 \;=\; (C' - C_0)V_0^2 \;=\; \frac{2\pi(\epsilon_r - 1)\epsilon_0 L V_0^2}{\ln(b/a)}$$

3. (4 pts) The battery is now disconnected. Shortly after, the cable is removed from the oil and the oil inside of it quickly drains (without removing any free charge on the conductors). What is the potential on the inner conductor after the oil is drained?

$Q' = C'V_0$ before the oil is drained. After the oil is drained, Q' remains the same => $Q' = C_0 V_{final}$

$$=> \quad V_{final} \;=\; \frac{C'}{C_0} V_0 \;=\; \epsilon_r V_0$$

4. (4 pts) Later, the cable is left lying horizontal, connected to the battery, exactly half filled with oil. Find $\vec{E}$ and $\vec{D}$ within the cable in both regions, $z > 0$ and $z < 0$.

$\int_a^b \vec{E} \cdot d\vec{r} = -V_0$ in both regions, therefore $\vec{E}$ is the same in both regions and is the same as we computed in part (1):

$$\vec{E} \;=\; \vec{E}(r) \;=\; \frac{Q_{enc}}{2\pi r \epsilon_0 L}\,\hat{r} \;=\; \frac{C_0 V_0}{2\pi r \epsilon_0 L}\,\hat{r} \;=\; \frac{V_0}{r \ln(b/a)}\,\hat{r}$$

$$\text{For} \quad z > 0, \quad \vec{D} \;=\; \epsilon_0 \vec{E}$$

$$\text{For} \quad z < 0, \quad \vec{D} \;=\; \epsilon_r \epsilon_0 \vec{E}$$

Part VI. (17 pts) **Dipoles and Images** A point dipole, $\vec{p}$, (such as a polar molecule) is placed a distance s above a large conducting plane. The dipole makes an angle, θ, relative to the normal vector to the plane (which we'll call the z axis).

1. (3 pts) What is the net charge that the dipole induces on the conducting plane? Explain.

We can think of a dipole as a positive charge that is separated by a small distance from an equal an opposite negative charge. We know that a charge q above a large conducting plate induces an image charge $-q$ on the plate. By superposition, a dipole above a large conducting plate induces equal and opposite positive and negative charges, making the net charge induced to be zero.

2. (6 pts) Use the method of images to find the potential along the z axis , $V(z)$, for $z > 0$. Express your answer in terms of p, θ, and s.

The potential of a dipole, $\vec{p}$, is given by $V(\vec{r}) = \vec{p} \cdot \hat{r}/(4\pi\epsilon_0 r^2)$. Treating the dipole as separate + and − charges lying along a line that makes an angle θ with respect to the z axis, the image of the + charge is a − charge equidistant from the plane and the image of the − charge is a + charge. The image dipole, $\vec{p}_{im}$, then lies at the point $-s$ on the z axis, making an angle, $-\theta$ with respect to the z axis.

For a point z on the $\hat{z}$ axis, $\vec{p} \cdot \hat{r} = p \cos\theta$ for $z > s$ and $-p\cos\theta$ for $z < s$, while $\vec{p}_{im} \cdot \hat{r} = p_{im} \cos\theta = p\cos\theta$ for both $z > s$ and $z < s$.

$$\text{Therefore} \quad V(z) = \frac{p\cos\theta}{4\pi\epsilon_0}\left[\pm\frac{1}{(s-z)^2} + \frac{1}{(s+z)^2}\right]$$

where we use the + sign for $z > s$ and the − sign for $0 < z < s$.

3. (5 pts) For $\theta = 0$, find the magnitude and direction of the force on $\vec{p}$.

The force on a dipole is $\vec{F} = (\vec{p} \cdot \vec{\nabla})\vec{E}$, where for this problem, $\vec{E} = \vec{E}_{im}$ is the electric field of the image dipole at the location of $\vec{p}$, and $\vec{p} \cdot \vec{\nabla} = p\, d/dz$ because $\theta = 0$.

For $\theta = 0$, the electric field of a dipole is $\vec{E}_{dip}(r, \theta = 0) = 2p\hat{r}/(4\pi\epsilon_0 r^3)$

$$=> \quad \vec{E}_{im} = \frac{2p\hat{z}}{4\pi\epsilon_0(s+z)^3} \quad => \quad \vec{F} = F_z\hat{z} = \frac{2p^2}{4\pi\epsilon_0}\frac{\partial}{\partial z}\left[\frac{1}{(s+z)^3}\right]_{z=s} = -\frac{6p^2}{4\pi\epsilon_0(2s)^4}\,\hat{z}$$

4. (5 pts) For $\theta = 45°$, what is the direction of the torque, $\vec{N}$, acting on $\vec{p}$?

Think of the dipole as a + and − charge separated by a small distance. For $\theta = 45°$, the + charge lies to the right of the $\hat{y}$ axis at a height greater than the − charge, while the image dipole has its − charge to the right of the $\hat{y}$ axis at a height lower then the image + charge. The original − charge and its image + charge are therefore closer to one another than the other pair of charges and exert a greater force on one another than the other pair. This attractive force is in the direction to cause the original dipole to rotate counterclockwise. The torque points out of the page.