Assume the atmosphere is transparent to the insolation (mainly visible and ultraviolet) so that the energy reaching the planet averaged over a day is

$$S_0\Lambda(x,t) \qquad (1)$$

where S_0 (the solar "constant") depends on solar luminosity and distance from the sun and is ~1370 W/m^2 today, and $\Lambda(x,t)$ takes into account the angle of incidence as a function of latitude (= arcsin(x)) which depends on time of day, time of year, and Milankovich components eccentricity, precession and obliquity.

In the simplest case where all energy arriving at the earth's surface is absorbed, the absorbed energy is that energy hitting the disk cross section (shadow area) of the earth

$$S_0\pi r^2 \qquad (2)$$

where r is the radius of the planet.

A fraction of the incident energy is reflected back to space. (This fraction is called the albedo (α). The planetary albedo is generally taken to be 0.3.) So we must modify (2) to take this into account; hence energy absorbed is

$$S_0\pi r^2(1-\alpha) \qquad (3)$$

In equilibrium, the net energy absorbed must be balanced by the net energy lost to space. Assuming that the earth is a perfect blackbody, the radiation to space is

$$\sigma T_g^{\,4}\left(4\pi r^2\right) \qquad (4)$$

where T_g is the ground temperature (annual, globally averaged) and the Stephan-Boltzman constant σ is 5.67 * 10^{-8} Wm^{-2}K^{-1}.

The entire energy balance for an atmosphere-free earth is therefore

$$\sigma T_g^{\,4} = \frac{S_0}{4}(1-\alpha)$$

Approximate the atmosphere as a thin radiating slab with temperature T_a and with a selection of greenhouse gases such at the net emission ε is some fraction of that of a blackbody at the same temperature. The net energy emission from the atmosphere is therefore

$$\varepsilon\sigma T_a^{\,4}\left(4\pi r^2\right)$$

where ε is some "bulk emissivity".

Taking into account the fact that the slab atmosphere will radiate to both to space and back down to earth, we can now write the full energy balance for the ground temperature, for a "realistic" earth with greenhouse gases:

$$\begin{aligned} &S_0\pi r^2(1-\alpha) + 4\pi r^2\varepsilon\sigma T_a^4 = 4\pi r^2\sigma T_g^{\,4} \\ &\frac{S_0(1-\alpha)}{4\sigma} + \varepsilon T_a^4 = T_g^4 \end{aligned} \tag{5}$$

The atmosphere in our model gets no direct energy from the sun, and recognizing that partial emitters are partial absorbers, the atmosphere gets only some of the energy emitted by the ground, so that the energy balance for the atmosphere requires that

$$\begin{aligned} &-2\varepsilon\sigma T_a^4 + \sigma\varepsilon T_g^{\,4} = 0 \\ &T_g^4 = 2T_a^4 \end{aligned} \tag{6}$$

So the solution for the ground temperature is:

$$T_g^4 = \frac{S_0(1-\alpha)}{4\sigma\left(1-\varepsilon/2\right)} \tag{7}$$

Without an absorbing atmosphere, the ground temperature would be

$$T_g = \sqrt[4]{\frac{S_0(1-\alpha)}{4\sigma}} = \text{ 255 K } = \text{ -18 C} \tag{8}$$

The increase in surface temperature due to greenhouse gases is therefore

$$\frac{T_g}{T_g(\varepsilon =)} = \sqrt[4]{\left(1-\varepsilon\!/\!2\right)} \tag{9}$$

The observed ground temperature is obtained using ε = -.76.

Climate sensitivity is defined as a climate change due to some external forcing, Z.

$$\lambda \equiv \frac{dT_g}{dZ} = \frac{\partial T_g}{\partial Z} + \sum_j \frac{\partial T_g}{\partial y_j}\frac{\partial y_i}{\partial z} \tag{10}$$

where y_j are variables that are parameterically related to Z.
Equations (11), (12), (13) (you solve 'em!) give sensitivity to various forcings (.e.g. changing S or ε).

We can make the EBM time dependent as follows:

$$C_a \frac{\partial T_a}{\partial t} = \varepsilon\sigma T_g - 2\varepsilon\sigma T_a{}^4 \tag{14}$$

$$C_g \frac{\partial T_g}{\partial t} = \frac{S_0(1-a)}{4} + \varepsilon\sigma T_a^4 - \sigma T_g^4 \tag{15}$$

where the left hand sides show the rate of change of energy storage in the atmosphere (14) and ocean (15) (treating the earth to first order as all "ground" is ocean).

To simplify, these equations can be linearized about a reference temperature $T_r = 273.15$.

$$T_a = T_r + T_a' \tag{16}$$
$$T_g = T_r + T_g' \tag{17}$$

so that (14) and (15) become

$$C_a \frac{\partial T_a'}{\partial t} = -\varepsilon a + \varepsilon b T_g' - 2b\varepsilon b T_a' \tag{18}$$
$$C_g \frac{\partial T_g'}{\partial t} = \frac{S_0(1-\alpha)}{4} - a(1-\varepsilon) + \varepsilon b T_a' - bT_g' \tag{19}$$

where $a = \sigma T_r^4$ and $b = {}^{4a}\!/_{T_r}$

a = 316 W/m^2 and b = 4.6 W/m^{-2}K^{-1}

The heat capacity of the atmosphere is

$$C_a \equiv C_p^a \frac{\Delta p}{g} = 10^3\,\mathrm{Jkg^{-1}k^{-1}}10^4\,\mathrm{kgm^{-2}} = 10^7\,\mathrm{Jm^{-2}k^{-1}}$$

Let the heat capacity of the ocean approximate the heat capacity of the surface ("ground"). h = depth of the mixed layer = 75 m.

$$C_g \equiv C_{ocean} = C_p^{ocean} h\rho_{H2O} = 4*10^3*75*10^3 = 3*10^8\,\mathrm{m^{-2}k^{-1}}$$

So Cg/Ca in (18) and (19) = 30. Note that on the timescale of the annual cycle, the heat capacity of land (as opposed to ocean) is about C_a/4. The point is that over the ocean, Cg = Co, and because Co>>Ca over the ocean we can assume the atmosphere is in equilibrium with respect to changes in the "ground" (sea surface) temperature. This allows us to set the left-hand side of (18) to zero and obtain:

$$2\varepsilon b T_a' = \varepsilon b T_g' - \varepsilon a \tag{20}$$

and the ground temperature (that is, the sea surface temperture) equation simplifies to

$$C_o \frac{\partial T_g'}{\partial t} = \frac{S(1-\alpha)}{4} - A - BT_g' \tag{21}$$

where A = a(1-ε/2)=195 W/m^2 and B = b(1- ε/2) = 2.9 Wm^{-2}K^{-1}.

The equilibrium solution to (20) and (21) is

$$T_g' = \left(\frac{1370}{4}(1-.3) - 195)/2.9\right) = 15.4\ C$$

It is useful to rearranging 21 to define the forcing, F as

$$F \equiv \frac{S(1-\alpha)}{4} - A = C_o \frac{\partial T_g'}{\partial t} + BT_g' \tag{22}$$

which has the time-dependent solution

$$T' = e^{-t/\tau} \int_0^t \frac{F(\mathrm{t})}{C_o} e^{\mathrm{t}/\tau} d\mathrm{t} \tag{23}$$

where

$$\tau = \mathrm{C_o/B} \tag{24}$$

is the characteristic response (adjustment) time of the system. This is about 3.3 years for thermal changes due to forcing of the upper ocean-atmosphere system. Over land, it is < 2 weeks.

For an instantaneous switch-on of a constant forcing,

$$F = \begin{cases} 0 & t \le 0 \\ F_o & t > 0 \end{cases} \qquad T_g'(t) = \frac{F_o}{B}\left(1 - e^{-t/\tau}\right) \tag{25}$$