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Considering Risk Taking Behaviors in Second-Best Toll Pricing

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1 Abstract

In this paper, we study the second-best toll pricing problem that aims to better distribute traffic by imposing tolls on a selected set of links in a network. We show that if the user equilibrium solution is not unique, many previous toll pricing methods are risk-prone; particularly, one hopes that the realized user equilibrium solution, after the "optimal" toll is imposed, is the same as what was predicted by the model. However, from a robust design point of view, a better toll may be obtained by optimizing over the worst case scenario when the user equilibrium solution varies. We therefore propose a risk-averse second-best toll pricing approach, in contrast to the risk-prone approach by previous researchers. Two examples are provided in this paper to illustrate the risk-prone and risk-averse toll pricing schemes for static and dynamic second-best toll pricing respectively.

2 Introduction

The Second-Best Toll Pricing (SBTP) is to determine the optimal toll scheme for a given set of links in a transportation network so that the traffic can be distributed more efficiently from the system point of view. SBTP can be categorized as static SBTP and dynamic SBTP (DSBTP), depending on whether traffic dynamics are considered or not. In this paper, we focus on the static SBTP (which is referred to as SBTP thereafter in this paper). However, we provide a small example on DSBTP in Section 5 as well.

SBTP research in the literature is rich and still growing. Many researchers have modeled SBTP as bi-level problems or MPECs (mathematical programs with equilibrium constraints)[1], [2], [3], [4]. The upper level is to optimize certain objective function from the transportation system's point of view and the lower level is a user equilibrium (UE) problem to account for the route choice behavior of individual motorists. Therefore, most existing SBTP methods aim to find "optimal" tolls (i.e., the so-called "upper level" decision variables) for the selected set of links so that the upper level objective can be minimized or maximized. Meanwhile, by solving the bi-level models, the lower level UE solution associated with the optimal tolls can also be obtained (which is denoted as "predicted" UE solution in this paper). Hence the bi-level model implicitly assumes that by applying the obtained toll pricing scheme, the resulting UE flow pattern is exactly what is predicted by the model and thus the desired system objective can be achieved. This premise is true if the UE solution is unique. However, if one relaxes the uniqueness assumption, the lower-level UE may have multiple solutions. More importantly, once the tolls are imposed, there is no other way in the context of toll pricing that one can "enforce" how drivers make their route choice decisions. Therefore, it is very possible that the "realized" UE flow pattern may deviate from what is predicted. If this is the case, the desired "optimal" objective may not be achieved and the designed toll scheme may not be effective.

The non-uniqueness of UE solutions, hence, represents some uncertainty in the SBTP design, which is not fully recognized in the literature. More specifically, many existing SBTP models assume, explicitly or implicitly, that the realized UE solution is equivalent to the "predicted" UE solution. In this paper, we show that these modeling methods for SBTP are "risk prone", i.e., the toll obtained from these methods is designed in such a way that one "hopes for the best thing to happen." However, since one has no control which UE solution will be realized given a toll setting, a more appropriate toll pricing scheme should be able to account for this uncertainty. In particular, from a robust design point of view, one may want to design the toll so that it is optimal for the "worst-case scenario" while the UE solution varies. This corresponds to the "risk-averse" approach.

In this paper, we focus on SBTP when the solution of the underlying UE problem is not unique. We start with a well-known fact that if the link cost function is only monotone (instead of strictly monotone), the solution set of a UE is not a singleton but a nonempty, compact and convex set. In other words, the UE solution is a set-valued map of the toll variable, rather than a one-to-one mapping (which is the case if the UE has a unique solution). We show that in this case existing bi-level "risk-prone" SBTP models can be converted into a single-level problem, which is defined on the graph of the set-valued map. To account for the non-uniqueness of the UE solution, we adopt the robust optimization technique [5]. In particular, from the robust design perspective, the implemented toll should be optimal for the worst case scenario while the UE solution varies. Following this concept, SBTP can be formulated as a "min-max" problem that corresponds to a "risk-averse" design approach. We then provide a small SBTP example to illustrate the results of risk-prone and risk-averse toll pricing approaches.

Most research efforts on SBTP so far are for static cases. Dynamic SBTP (DSBTP) or time-varying tolls, however, has received more attention recently [6], [7], [8], [9]. In this paper, we show via another small example that for DSBTP, risk-taking behaviors may also need to be considered. This is particularly true since the solution for dynamic user equilibria (DUE) on which DSBTP is built may be multi-valued as well.

This paper is organized as follows. Section 3 focuses on the derivations of the risk-prone and risk-averse toll pricing approaches. It starts with the introduction of the VI-based UE model and then discusses the risk-prone SBTP model that can be formulated as an MPEC. The risk-averse model is also developed, which is formulated as a "min-max" problem. An illustrate example is provided in Section 4. We show that at least for this small example, the risk-averse approach is superior to the risk-prone approach. In Section 5, an illustrative example for DSBTP is provided. We show that if the DUE solution is not unique, risk-taking behaviors can be considered for DSBTP in a similar fashion as that for the static case. Some concluding remarks and future study directions are given in Section 6.

3 Risk-Prone and Risk-Averse SBTP Approaches

This section discusses the risk-prone and risk-averse SBTP design approaches. The discussion starts with the VI-based UE model.

3.1 VI-Based UE Model

Assume a traffic network can be represented as a directed graph $G(N, A)$ where N is the set of nodes and A is the set of links. In this paper, we use index i or j to denote a node, and index a to denote a link. Denote x_a the total traffic flow on link a and $x = (x_a)_{a \in A}$ the vector of link flows. Further assume $t(x)$ is the link travel time function defined on total link flow vector x. Then the "baseline" UE model, denoted as *UE(0)*, can be formulated as follows:

$$\textit{UE(0)} \qquad t(x)^T(q - x) \geq 0, \forall q \in K. \tag{1}$$

Here K denotes the feasible set of link flow, which is a nonempty, compact, and convex set. Note that "0" in the parenthesis in $UE(0)$ indicates that there is no toll imposed (the base case). It is well-known that if t is strictly monotone on K, then $UE(0)$ has a unique solution (provided the problem is feasible which is usually the case).

However, if t is only monotone (or even pseudo-monotone), $UE(0)$ may not have a unique solution. Denote the solution set of $UE(0)$ as $S(0)$. Facchinei and Pang [10] showed that in this case, $S(0)$ is nonempty, compact, and convex. This is formally stated in the following Lemma. Notice that since this paper is mainly to propose and illustrate the two toll design approaches, we provide several lemmas and theories for this purpose without detailed proofs.

Lemma 1 *If t is continuous and monotone and K is nonempty, compact, and convex, then $UE(0)$ has a nonempty, compact and convex solution set.*

Assume y is a given vector and $UE(y)$ denotes the UE problem parameterized by y which is defined as:

$$\textit{UE(y)} \qquad (t(x) + y/\theta)^T(q - x) \geq 0, \forall q \in K. \tag{2}$$

Here θ denotes the "value of time" so that $t(x) + y/\theta$ in (2) is the generalized travel time by considering tolls, denoted as $c(x, y)$. That is,

$$c(x, y) = t(x) + y/\theta. \tag{3}$$

Denote $S(y)$ as the solution set of $U(y)$, we have the following Lemma.

Lemma 2 *If t is continuous and monotone and K is nonempty, compact, and convex, then $UE(y)$ has a nonempty, compact and convex solution set for any given and finite vector y.*

3.2 Risk Prone SBTP - the MPEC Model

Denote y in $UE(y)$ as the toll vector imposed on a traffic network. Since we are dealing with SBTP, we set $y_a = 0, \forall a \in A \setminus P$, where P is the set of tolled links. Traditionally the SBTP can be formulated as the following MPEC, denoted as $MPECSBTP$:

$$
\begin{aligned}
MPECSBTP \qquad & \min_{y,x(y)} \quad f(y, x(y)) && (4)\\
& \text{s.t.} \quad y \in K_y && (5)\\
& \qquad x(y) \quad \text{solves UE}(y). && (6)
\end{aligned}
$$

Here $f(y, x(y))$ denotes the objective function for toll design, which could be, e.g., the total system travel time. It is assumed to be a function of both the toll vector y and the total link flow vector x. The latter is itself a function of y. Further $K_y = \{y | y_l \leq y \leq y_u\}$ is the bound constraint of y and y_l and y_u are the lower and upper bounds of y respectively.

Suppose y^* is the optimal solution to *MPECSBTP* in terms of y. If $UE(y)$ has a unique solution x^* at y^*, one can expect that if the optimal toll pricing y^* is imposed, the realized UE state will be exactly x^*. In other words, the optimal system objective will be achieved as one would expect.

However, as stated in Lemma 2, if t is only monotone with respect to x, there may be multiple UE solutions given the optimal toll vector y^*. In this case, the optimal UE solution associated with y^* in solving $MPECSBTP$ (i.e., x^*) is called the "predicted" UE solution. Clearly $x^* \in S(y^*)$. However, once the optimal toll y^* is imposed, one has no control which UE solution in $S(y^*)$ will be realized. Therefore, the optimal upper level objective value may not be achieved (or can only be achieved if the predicted UE solution x^* is realized). In this sense, the $MPECSBTP$ model represents a "risk-prone" approach to design tolls: one "hopes" that the predicted UE solution could be obtained when the optimal toll is imposed.

In case the UE solution is not unique, the solution set $S(y)$ is a set-valued map of the toll vector y (see [10]). The $MPECSBTP$ model can thus be converted into a single level problem as follow:

$$
\begin{aligned}
SLSBTP \qquad & \min_{y,x} \quad f(y, x) && (7)\\
& \text{s.t.} \quad (y, x) \in G. && (8)
\end{aligned}
$$

Here
$$G = \{(y, x) \mid x \in S(y), \quad y \in K_y\}$$
is the graph of the set-valued map $S(y)$. Denote the above model as $SLSBTP$ which stands for "Single-Level SBTP."

The following Theorem first states that the graph G is compact, based on which the solution existence condition for *SLSBTP* is established.

Theorem 1 *If t is continuous and monotone with respect to x, $f(y, x)$ is continous with respect to (y, x), and K is nonempty, compact, and convex, then the following two statements hold.*

(a) G is compact; and

(b) SLSBTP has at least one solution.

3.3 Risk Averse SBTP - The Min-Max Model

As aforementioned, once the designated toll (such as from solving *MPECSBTP* or *SLSBTP* in Section 3.2) is imposed, one has no control on how individual motorists will make their route choice decisions. Therefore, if the UE solution is not unique, it will be uncertain which UE flow pattern (solution) will be realized once the toll is imposed. Further, some of the realized UE solutions may degrade the system objective in the upper level. Therefore, the toll design needs to account for this uncertainty so that it is optimal no matter how the UE solution varies.

In this paper, we adopt the robust optimization concept and the optimal toll can be designed in such a way that it is optimal for the worst case scenarios. This represents a "risk-averse" approach for toll pricing, which can be expressed as a "min-max" problem (denoted as *MMSBTP*) as follows:

$$MMSBTP \qquad \min_{y \in K_y} \quad \max_{x \in S(y)} \quad f(y,x) \tag{9}$$

We can see that *MMSBTP* aims to minimize, as y varies within K_y, the largest objective function $f(y,x)$ over all x's in $S(y)$. Assume y^* the computed optimal toll by the risk-averse model and $x^* \in S(y^*)$ its associated UE solution. We will have $f(y^*,x^*) \leq \max_{\forall x \in S(y)} f(y,x), \forall y \in K_y$. In other words, the risk-averse approach generates a solution that is optimal for the worst-case scenario (i.e., the largest possible objective value over the UE solution set is minimized).

Clearly there are other ways to account for uncertainties besides the robust optimization technique, such as stochastic programming [11]. However, the latter requires knowledge of the distribution of the uncertainty which is hard to achieve in our case. That is, given a toll scheme, which UE flow pattern will be realized is completely random; yet, it also seems unrealistic to assume the probability of realization of a certain UE flow pattern (It may seem intuitive to assume the probability of a specific solution is uniform across all possible solutions. However, for a general convex solution set, an explicit representation for such a distribution may not be trivially obtained.). Hence, we argue that robust optimization may more appropriately capture this type of uncertainty compared with stochastic programming based approaches.

Note that here we assume that *MMSBTP* has at least one solution, which may not be necessarily true. In case such a solution does not exist, *MMSBTP* can be replaced by the following "inf-max" model instead:

$$IMSBTP \qquad \inf_{y \in K_y} \quad \max_{x \in S(y)} \quad f(y,x) \tag{10}$$

The following theorem shows that the optimal objective function value of *IMSBTP* exists; in other words, the values

$$\max_{x \in S(y)} \quad f(y,x)$$

have a greatest lower bound as y varies within K_y.

Theorem 2 *If t is continuous and monotone with respect to x, $f(y,x)$ is continous with respect to (y,x), and K is nonempty, compact and convex, then*

$$\inf_{y \in K_y} \quad \max_{x \in S(y)} \quad f(y,x) > -\infty. \tag{11}$$

Hence, although the solution of *IMSBTP* may not be obtained on K_y, its optimal objective value does exist (as shown in Theorem 2). This objective value provides the greatest lower bound of the objective value of *MMSBTP* and thus valuable information for SBTP design.

4 An Illustrative Example

To better illustrate the risk-prone and risk-averse design approaches, we provide a small example in this section.

Figure 1 depicts a hypothetical network with one origin-destination (OD) pair (from node r to node s) and three routes. A toll booth is located at the very beginning of route 2 and 3. The distance between node r and i is very small so that the travel time can be ignored (assume toll is automatically collected and therefore the delay at the toll booth can be ignored as well). Further assume the total demand $d = 10$ and the route (also link) flow are x_1, x_2, and x_3. For simplicity, we assume the "value of time" $\theta = 1$. Then the link generalized travel times, with toll imposed, are assumed to be:

$$\begin{aligned} c_1 &= \ 2x_1 + x_2 + x_3 \\ c_2 &= \ 2x_2 + 2x_3 + y \\ c_3 &= \ 2x_2 + 2x_3 + y. \end{aligned}$$

Here y is the toll and $y \in K_y = \{y|0 \leq y \leq 10\}$. Denote $c = (c_1, c_2, c_3)^T$ and $x = (x_1, x_2, x_3)^T$. It is easy to observe that c is monotone, but not strictly monotone, with respect to x. To see this, we note that the Jacobian matrix of c over x is

$$J = \partial c/\partial x = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix}.$$

Clearly, J is not symmetric and we have

$$(J + J^T)/2 = \begin{bmatrix} 2 & 0.5 & 0.5 \\ 0.5 & 2 & 2 \\ 0.5 & 2 & 2 \end{bmatrix},$$

which is symmetric and positive semi-definite, but not positive definite. Therefore, c is monotone with respect to x, but not strictly monotone.

INSERT FIGURE 1 HERE

We first look at the solution set of $UE(y)$, i.e., $S(y)$ for any given $y \in K_y$. Since we have $c_2 = c_3$, there are three cases that we need to consider: i) only route 1 carries flow, ii) only routes 2 and 3 carry flow, and iii) all three routes carry flow. For case i), we have $x_1 = 10, x_2 = x_3 = 0$. This leads to $c_1 = 20 > c_2 = c_3 = y \leq 10$. Therefore, case i) is impossible. For case ii), we have $x_1 = 0, x_2 + x_3 = 10$. This leads to $c_1 = 10 < c_2 = c_3 = 20 + y \geq 20$, which is also impossible. Therefore, all three routes must carry flow and we have $c_1 = c_2 = c_3$. This gives us

$$S(y) = \{x = (x_1, x_2, x_3)^T \geq 0 | x_1 = (10 + y)/3, \quad x_2 + x_3 = (20 - y)/3\}. \tag{12}$$

Clearly, for any given $y \in K_y$, $S(y)$ is a straight line (i.e., a nonempty polyhedral set)in the three dimension space $x_1 - x_2 - x_3$ as shown in Figure 2.

INSERT FIGURE 2 HERE

To determine the "optimal" toll, we first assume the objective function for the upper level as follows:

$$f(y,x) = t_1 x_1 + 3t_2 x_2 + t_3 x_3. \tag{13}$$

Note that in the above definition, we assign different weights to different links (routes). In particular, the weight of link 2 is set as 3. This may be appropriate if route 2 goes through an area which is more adversely impacted by traffic (in terms of vehicle-miles-traveled) than other areas.

Given the above, the risk-prone approach (i.e., the current SBTP practice) is to solve $SLSBTP$. First, since $S(y)$ can be explicitly expressed in equation (12) for a given y, the upper level objective function $f(y,x)$ can be rewritten as:

$$f(y,x) = (40+y)(10+2x_2)/3. \tag{14}$$

Obviously, for a fixed $y \in K_y$, $f(y,x)$ is minimized when $x_2 = 0$. Actually, $x_1 = (10+y)/3, x_2 = 0, x_3 = (20-y)/3$ is the unique and global minimizer of $f(y,x)$ when y is given since $f(y,x)$ in (14) is linear for fixed y. This minimizer is the intersecting point of $S(y)$ and the $x_1 - x_3$ plane. Therefore, as y varies from 0 to 10, the trajectory of minimizers of $f(y,x)$ is the line on the $x_1 - x_3$ plane, as shown in Figure 2.

To find the solution to $SLSBTP$, therefore, we need to solve the following problem:

$$\min_{y \in K_y} \quad f(y) = 10(40+y)/3. \tag{15}$$

This is a linear programming problem and we have $y_p^* = 0$ the optimal solution. Here the subscript "p" denotes "risk-prone". The predicted UE solution is $x_{p,1}^* = 10/3, x_{p,2}^* = 0, x_{p,3}^* = 20/3$ and the associated objective value is $z_p^* = 400/3$.

Most existing SBTP design methods will stop here with the above solution, which simply states that no toll should be implemented. However, as the UE solution at the computed "best" toll $y_p^* = 0$ is not unique, the realized UE solution can be any point in $S(0)$, which is the line as shown in Figure 2. In particular, if the realized UE solution is on the $x_1 - x_2$ plane ($x_1 = 10/3, x_2 = 20/3, x_3 = 0$), the objective value will be much higher as $\hat{z}_p^* = 2800/9$. As shown later, $\hat{z}_p^*$ is actually the highest possible objective value (the worst case) when $y = 0$. This clearly illustrates that the risk-prone approach is not reliable when the UE solution is not unique.

For the "risk-averse" approach, we first find the maximizer of $f(y,x)$ for a given y. This is equivalent to maximize $f(y,x)$ in equation (14) over set $S(y)$. Clearly, this is achieved when x_2 is maximized at $x_2 = (20-y)/3$. Thus the unique and global maximizer of $f(y,x)$ for a given y is $x_1 = (10+y)/2, x_2 = (20-y)/3, x_3 = 0$, which is at the $x_1 - x_2$ plane.

Next, substitute $x_2 = (20-y)/3$ to equation (14), we obtain the objective value for a given y for the averse case as:

$$f(y) = (40+y)(70-2y)/9. \tag{16}$$

Note that (16) is equivalent to the inner maximization problem in (9). Hence, to find the optimal solution of the risk-averse approach, one needs to minimize (16) over K_y. This can be easily solved with a unique and global solution $y_a^* = 10$. Here the subscript "a" denotes "risk-averse". The predicted UE solution is $x_{a,1}^* = 20/3, x_{a,2}^* = 10/3, x_{a,3}^* = 0$ and the associated objective value is $z_a^* = 2500/9$. Note that this value is less than that of the worst case scenario by the risk-prone toll scheme $y_p^* = 0$ (i.e., $\hat{z}_p^*$).

Distinct from the risk-prone approach, the upper level objective value will decrease as the realized UE solution varies under the risk-averse optimal toll $y_a^* = 10$. In particular, if the UE solution on the $x_1 - x_3$ plane (i.e., $x_1 = 20/3, x_2 = 0, x_3 = 10/3$) is realized at y_a^*, the objective value is $\hat{z}_a^* = 500/3$. Since this UE solution is at the $x_1 - x_3$ plane, the resulting objective value is the lowest possible objective value (the best case) when $y = 10$.

To further compare the performance of the two toll pricing approaches, we compute the average value (the average of the best and worst scenarios) and variation (the difference of the best and worst scenarios) of the upper level objective value for a given toll. This is shown in Figure 3. The risk-prone and risk-averse solutions have the same average value: 2000/9. The variation for the risk-prone approach is $|\hat{z}_p^* - z_p^*| = 2800/9 - 400/3 = 1600/9$, which is higher than that for the risk-averse approach: $|\hat{z}_a^* - z_a^*| = 2500/9 - 500/3 = 1000/9$. Clearly, $y = 10$ generates a set of solutions whose average objective value is the same as those by $y = 0$, but with a smaller variation. Therefore, at least for this small example, we may say that the risk-averse design approach is superior to the risk-prone approach (which is currently the most popularly used approach for SBTP).

INSERT FIGURE 3 HERE

5 An Example of DSBTP

Risk taking behaviors may also need to be considered when designing DSBTP. Since DSBTP relies on dynamic user equilibria (DUE) which is more complicated than static UE, it is expected that the uniqueness of the time-dependent link flow or inflow requires stronger conditions. On the other hand, if the time-dependent link flow is not unique, the risk-prone and risk-averse toll pricing approaches can be defined similarly as in Sections 3.2 and 3.3. In this section, instead of investigating the detailed formulation of DSBTP (which may be found in [8] and [6]), we illustrate the two toll pricing approaches using a small example.

Figure 4 shows a simple network with one OD pair (from origin r to destination s) and two connecting links, Link 1 and Link 2. The free flow travel times of Link 1 and Link 2 are $\tau_{01} = 5$ and $\tau_{02} = 10$, respectively. The capacities of these two links are $C_1 = 20$ and $C_2 = 50$. We assume the demand from r to s is $d = 60$ for a period of time $T = 30$.

INSERT FIGURE 4 HERE

A toll booth is located on Link 1 to control the traffic going through this link to the destination. For illustrative purposes, we consider a special case of DSBTP: traffic dynamics of the network is considered, while we assume the toll on Link 1 (denoted as y) is fixed during the entire period T. We further assume $0 \le y \le 10$ and the "value of time" parameter $\theta = 1$.

We first look at the DUE solution of the network, which can be obtained via constructing the queueing diagrams of the two links [7], [12]. It turns out that we have three cases, depending on the value of y: $0 \le y < 5$, $5 < y \le 10$, and $y = 5$. Figure 5 depicts the queueing diagrams for $0 \le y < 5$. The upper plot is for Link 1 and the lower one is for Link 2. The vertical axis of both plots is the accumulative number of vehicles that have entered or left the link, while the horizontal axis is time. Assume that vehicles start to enter the simple network at $t = 0$. The bold solid lines in this figure represent the entrance or departure curves to both links. In this paper, we assume that although congestion may happen on both links, there is sufficient storage so that traffic will not spill over to the origin r. Then it is easy to see that when $0 \leq y < 5$, all demands will be assigned to Link 1 first because $\tau_{02} > \tau_{01} + y$ and we assume motorists have perfect information. Since the capacity of Link 1 is much smaller than the total demands, congestion will build on Link 1 and its travel time will increase. However, all traffic will still be assigned to Link 1 until the travel time of Link 1 reaches $\tau_{02} - y$. Denote the time when this happens is t_{01}. Then from 0 to t_{01}, the DUE solution, i.e., the time-dependent inflow rates to the two links, is $x_1(t) = d$ and $x_2(t) = 0$, $\forall 0 \leq t \leq t_{01}$. After t_{01}, the inflow rate of Link 1 reduces to $x_1(t) = C_1$ and that of Link 2 increases to $x_2(t) = d - C_1$, $\forall t_{01} \leq t \leq T$. Therefore, the DUE solution for $0 \leq y < 5$ is unique.

INSERT FIGURE 5 HERE

For $5 < y \leq 10$, we can apply the similar analysis as shown in Figure 6. In this case, Link 2 will be selected first. One can see the DUE solution is also unique.

INSERT FIGURE 6 HERE

For $y = 5$, traffic will be assigned to both links at the very beginning. More importantly, since $C_1 + C_2 > d$, the DUE solution is not unique. As shown in Figure 7, the bold solid lines represent the solution

$$x_1(t) = C_1, x_2(t) = d - C_1, \forall 0 \leq t \leq T.$$

The bold dash lines, on the other hand, illustrate another solution

$$x_1(t) = d - C_2, x_2(t) = C_2, \forall 0 \leq t \leq T.$$

In fact, these two solutions represent two extreme cases; any solution in between them is also an optimal DUE solution. That is to say, any dynamic flow that satisfies

$$d - C_2 \leq x_1(t) \leq C_1, x_2(t) = d - x_1(t)$$

is an optimal DUE solution for $y = 5$.

INSERT FIGURE 7 HERE

We next look at the DSBTP solution. In this paper, we assume that the objective for DSBTP is to minimize the total system travel time [6], which can be expressed as follows:

$$f(y, x) = \int_0^T (x_1(t)\tau_1(t) + x_2(t)\tau_2(t))dt. \tag{17}$$

Here $\tau_1(t)$ and $\tau_2(t)$ are the travel times of Link 1 and Link 2 at time t, respectively.

For $0 \leq y < 5$, the above objective function, denoted as $F_1(y)$, is also the summation of the three areas, marked as "1", "2", and "3" in Figure 5 between the entrance and departure curves of Link 1 and Link 2. We thus have:

$$F_1(y) = f_1 + f_2 + f_3, \tag{18}$$

$$f_1 = \frac{(\tau_{01} + \tau_{02} - y)t_{01}d}{2}, \tag{19}$$

$$f_2 = (\tau_{02} - y)(T - t_{01})C_1, \tag{20}$$

$$f_3 = (T - t_{01})(d - C_1)\tau_{02}. \tag{21}$$

Substitute f_1, f_2, f_3 into $F_1(y)$, we obtain:

$$F_1(y) = \frac{(\tau_{01} + \tau_{02} - y)t_{01}d}{2} - (T - t_{01})yC_1 + (T - t_{01})\tau_{02}d. \tag{22}$$

And from Figure 5, t_{01} can be computed as

$$t_{01} = \frac{(\tau_{02} - \tau_{01} - y)C_1}{d - C_1}. \tag{23}$$

Substitute equation (23) into (22), and noting that $\tau_{01} = 5, \tau_{02} = 10, C_1 = 20, C_2 = 50, d = 50$, we have:

$$F_1(y) = 5(y - 55)^2 + 2500. \tag{24}$$

For $5 < y \leq 10$, one can obtain the objective function in a similar fashion from Figure 6. This is denoted as $F_2(y)$:

$$F_2(y) = -100(y - 55/4)^2 + 24156.25. \tag{25}$$

Note that for any given $0 \leq y < 5$ or $5 < y \leq 10$, the objective value can be uniquely determined via equations (24) or (25) since the DUE solution is unique.

The objective function for $y = 5$, denoted as $F_3(y, x)$, can be calculated as:

$$F_3(y, x) = \int_0^T \left[x_1(t)\tau_{01} + (d - x_1(t))\tau_{02}\right] dt = \tau_{02}Td - \int_0^T (\tau_{02} - \tau_{01})x_1(t)dt. \tag{26}$$

Note that since the DUE solution is not unique for $y = 5$, the objective value is also a function of x as shown in (26). Further, since $\tau_{02} > \tau_{01}$, $F_3(y, x)$ is minimized when $x_1(t) = C_1$ and maximized when $x_1(t) = d - C_2$. Denote the minimum and maximum values of F_3 as $F_3^{\min}$ and $F_3^{\max}$, respectively. We have:

$$F_3^{\min} = \tau_{02}Td - (\tau_{02} - \tau_{01})C_1T = 15000, \tag{27}$$

$$F_3^{\max} = \tau_{02}Td - (\tau_{02} - \tau_{01})(d - C_2)T = 16500. \tag{28}$$

In summary, the DUE solution is unique for $0 \leq y < 5$ and $5 < y \leq 10$, but not unique for $y = 5$. For $y = 5$, however, there is a unique maximizer and minimizer for the objective function F_3. It is easy to check that $F_3^{\min}$ and $F_3^{\max}$ can also be achieved by substituting $y = 5$ into equations (24) and (25), respectively. In other words, the possible objective values for $0 \leq y \leq 10$ is actually continuous, but not differentiable due to the "kinks" at $y = 5$. This is shown in Figure 8.

INSERT FIGURE 8 HERE

As shown in the figure, the risk-prone solution can be obtained via minimizing the non-smooth curve in Figure 8 and we can observe that it is $y_p^* = 5$. To obtain the risk-averse solution, however, one needs to find the minimum of the largest possible objective values among all $0 \le y \le 10$. Due to the non-smoothness at $y = 5$, we can see that such a solution does not exist for this particular example. However, we can see that as y converges to 5 from the left, the objective value can be infinitely close to the minimum at $y = 5$, i.e., 15000. In practice, a toll that is smaller than but sufficiently close to 5 should work reasonably well.

This example illustrates the difference of the risk-prone and risk-averse toll pricing for DSBTP. Notice that DUE solution may be multi-valued even for small size networks, as shown in [12]. Furthermore, we show that the risk-averse solution may not exist in certain cases, depending on the actual network configuration and demand profiles. However, the lower bound of such a solution may exist, which could still provide insights on selecting proper dynamic tolls for a given network.

6 Conclusion

In this paper, we studied the risk-taking behaviors in second-best toll pricing (SBTP) when user equilibrium (UE) solution is not unique. We first argued that previous studies on SBTP assumed, explicitly or implicitly, the underlying UE solution is unique. We then relaxed the uniqueness assumption and showed that previous BSTP models are risk-prone in the sense that one hopes the realized UE solution, after the optimal toll is imposed, is exactly the same as what was predicted by the model. As a contrast, we proposed a risk-averse SBTP approach that can produce optimal tolls no matter how the UE solution may change (or the toll is optimal for the worst case scenario). We further presented two examples, one for the static SBTP and the other one for DSBTP, to illustrate the two toll pricing approaches.

The purpose of this paper is mainly to differentiate and define, further to illustrate, the risk-prone and risk-averse toll pricing approaches. The authors are now investigating the properties of the proposed risk-averse model and its solution algorithm. Results in this regard will be reported in subsequent papers. Also, we showed via the second example that in certain cases, the risk-averse solution may not exist for DSBTP. This may also happen to the static SBTP. This property and its implication to SBTP merits further investigations.

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List of Figures

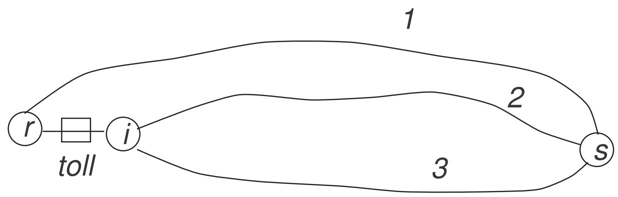

Figure 1: An Illustrative Example

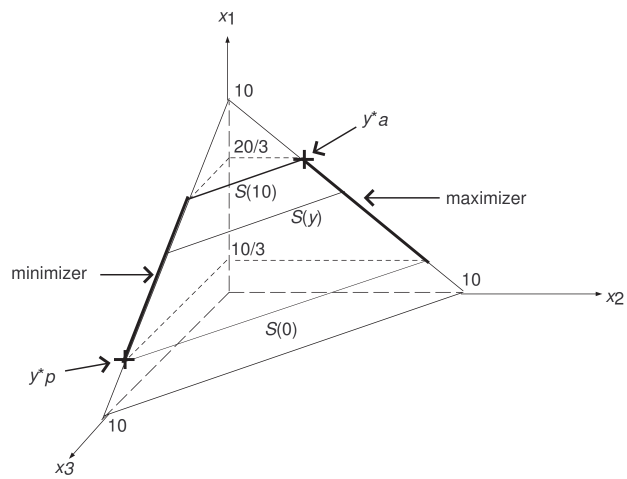

Figure 2: Solution Set of SBTP

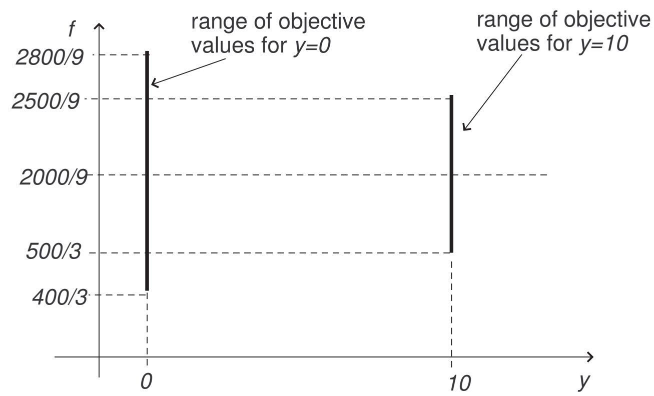

Figure 3: Comparison of Risk-Prone and Risk-Averse Solutions

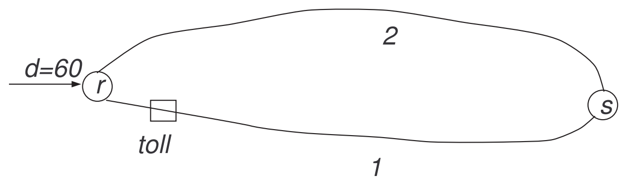

Figure 4: Example Network for Dynamic Second Best Toll Pricing

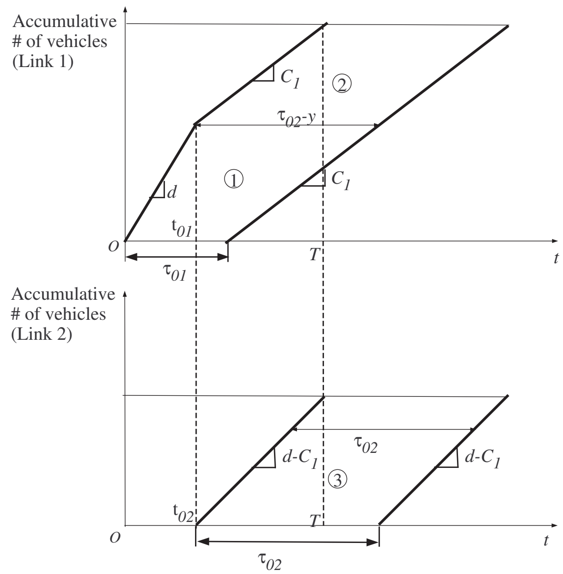

Figure 5: Queuing Diagram ($0 \leq y < 5$)

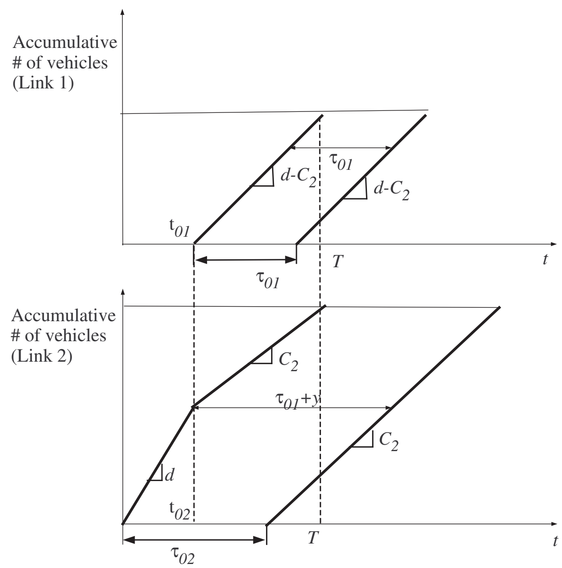

Figure 6: Queuing Diagram ($5 < y \leq 10$)

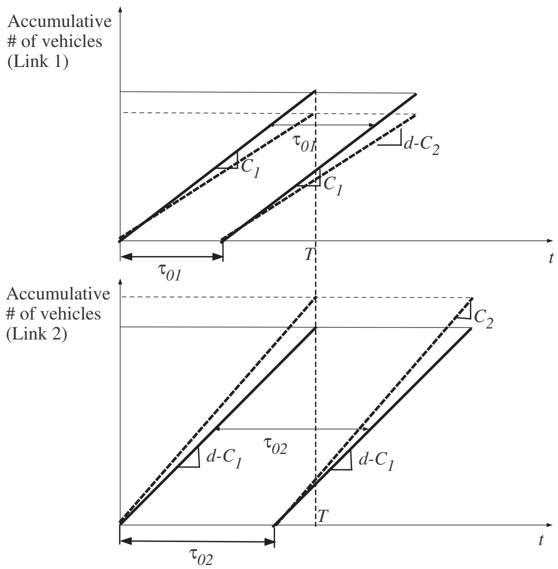

Figure 7: Queuing Diagram ($y = 5$)

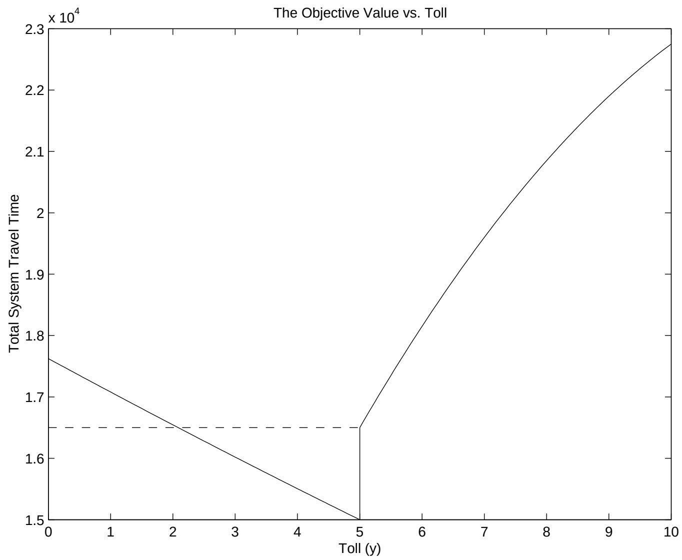

Figure 8: DSBTP Objective Values