GOVERNING EQUATIONS

I. REVIEW OF FLUID MECHANICS

- Shell balances can be performed on any property of interest (e.g. mass, momentum and energy)

- Consider a differential volume in cartesian coordinates and perform shell balances.

1) Shell balance on mass:

[Rate of flow of mass in] - [Rate of flow of mass out] + [Rate of generation] = [Rate of accumulation]

Assuming that there is no accumulation (steady state) or generation:

$$\Delta y\Delta z(\rho.V_x)|_x - \Delta y\Delta z(\rho.V_x)|_{x+\Delta x} + \Delta x\Delta z(\rho.V_y)|_y - \Delta x\Delta z(\rho.V_y)|_{y+\Delta y} + \Delta x\Delta y(\rho.V_z)|_z - \Delta x\Delta y(\rho.V_z)|_{z+\Delta z} = \Delta x\Delta y\Delta z.\frac{\partial\rho}{\partial t}$$

Dividing by the differential volume and taking the derivative as Δx, Δy, Δz -> 0 yields the *continuity equation*:

$$\rho\nabla\bullet\vec{V} + \vec{V}\bullet\nabla\rho + \frac{\partial\rho}{\partial t} = 0$$

If the fluid properties are constant, i.e. ρ is constant, then: $\nabla\bullet\vec{V} = 0$

Or in cartesian coordinates: $$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = 0$$ (steady state and constant properties)

2) Shell balance on momentum in x direction:

[Rate of x-momentum in by diffusion] - [Rate of x-momentum out by diffusion] + [Rate of x-momentum in by convection] - [Rate of x-momentum out by convection] = - [Sum of x-forces on shell] + [Rate of Accumulation]

Assuming no accumulation or generation and changing the signs:

$$\Delta y\Delta z\{\tau_{xx}|_{x+\Delta x} - \tau_{xx}|_x\} + \Delta x\Delta z\{\tau_{yx}|_{y+\Delta y} - \tau_{yx}|_y\} + \Delta x\Delta y\{\tau_{zx}|_{z+\Delta z} - \tau_{zx}|_z\} +$$
$$\Delta y\Delta z\{[V_x(\rho.V_x)]_{x+\Delta x} - [V_x(\rho.V_x)]_x\} + \Delta x\Delta z\{[V_y(\rho.V_x)]_{y+\Delta y} - [V_y(\rho.V_x)]_y\} +$$
$$\Delta x\Delta y\{[V_z(\rho.V_x)]_{z+\Delta z} - [V_z(\rho.V_x)]_z\} =$$
$$\Delta y\Delta z\{P|_{x+\Delta x} - P|_x\} + \Delta x\Delta y\Delta z(\rho.g_x) - \Delta x\Delta y\Delta z(\rho.\frac{\partial V_x}{\partial t})$$

Dividing by the volume, taking the limit as Δx, Δy, Δz -> 0 and substituting Newton's law of viscosity yields the *Navier-Stokes equation* in the x-direction:

$$\rho.\frac{DV_x}{Dt} = \rho.[\frac{\partial V_x}{\partial t} + \vec{V} \bullet \nabla V_x] = -\frac{\partial P}{\partial x} + \mu \nabla^2 V_x + \rho.g_x$$

At steady-state, constant properties (e.g. constant ρ and μ) the equation can be written:

$$\rho.(V_x.\frac{\partial V_x}{\partial x} + V_y.\frac{\partial V_x}{\partial y} + V_z.\frac{\partial V_x}{\partial z}) = -\frac{\partial P}{\partial x} + \mu.(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2}) + \rho.g_x$$

Similar expressions can be obtained in the y and z directions.
The equations of continuity and momentum can be used to obtain velocity distributions or calculate τ values which in turn can be used to determine friction coefficients.

II. SHELL BALANCE ON ENERGY IN CARTESIAN COORDINATES

Notations:

velocity:	$\vec{V} = V_x\vec{\delta_x} + V_y\vec{\delta_y} + V_z\vec{\delta_z}$ (written as $V_x\vec{i} + V_y\vec{j} + V_z\vec{k}$ in IDW)
magnitude of velocity:	$\vec{V} \bullet \vec{V} = V^2 = V_x^2 + V_y^2 + V_z^2$
internal energy per unit mass:	$\hat{U}$
kinetic energy per unit mass:	$\hat{K} = \frac{V^2}{2}$
Potential energy per unit mass:	$\hat{\phi}$
total energy per unit mass:	$\hat{E} = \hat{U} + \hat{K} + \hat{\phi}$

In an differential control volume, every point is at equilibrium and we can use thermodynamics relationships. For an open system at steady-state, we can assume that the energy is conserved and perform a shell balance:

[Rate of accumulation of total energy]	=	[Rate of energy in by advection]	-	[Rate of energy out by advection]	+	[Rate of energy in by conduction]	-	[Rate of energy out by conduction]	-	[Net rate of work done by system on sur.]	+	[source terms]
(1)		(2)				(3)				(4)+(5)		(6)

Term (1): $\Delta x \Delta y \Delta z. \frac{\partial(\rho\hat{E})}{\partial t} = \Delta x \Delta y \Delta z.\frac{\partial}{\partial t}\{\hat{U}\rho + \frac{V^2}{2}\rho + \hat{\phi}\rho\}$

Term (2): $\Delta y \Delta z\{[\rho V_x(\hat{U} + \frac{V^2}{2} + \hat{\phi})]_x - [\rho V_x(\hat{U} + \frac{V^2}{2} + \hat{\phi})]_{x+\Delta x}\}$ + corresponding terms in y and z

Term (3): $\Delta y \Delta z \{q''_x|_x - q''_x|_{x+\Delta x}\}$ + corresponding terms in y and z

Recall that work = (force).(distance in direction of the force)

Thus rate of work = (force).(velocity in the direction of the force)

Term (4): $\Delta y \Delta z \{PV_x|_{x+\Delta x} - PV_x|_x\}$ + y and z terms — rate of work against static pressure in x direction

Term (5): $\Delta y \Delta z \{[\tau_{xx}V_x + \tau_{xy}V_y + \tau_{xz}V_z]_{x+\Delta x} - [\tau_{xx}V_x + \tau_{xy}V_y + \tau_{xz}V_z]_x\}$ + y and z terms — rate of work against viscous forces in x direction

Note: changes in potential energy due to gravity are already accounted for in ϕ.

Term (6): $\Delta x \Delta y \Delta z . \dot{q}_g$

Now divide by $\Delta x \Delta y \Delta z$, take the limit as Δx, Δy, Δz -> 0, and rearrange using the equations of continuity and momentum. This leads to the *equation of energy*:

$$\rho . \frac{D}{Dt}(\hat{U} + \frac{V^2}{2} + \hat{\phi}) = (-\nabla \bullet \vec{q}) - (\nabla \bullet p\vec{V}) - \nabla \bullet [\vec{\vec{\tau}} \bullet \vec{V}]$$

Rate of gain of total energy per unit volume	Rate of energy input by conduction per unit volume	Reversible rate of energy input by compression per unit volume	Irreversible rate of energy change by viscous dissipation

Now substitute:

- Thermodynamics relationships: $h = U + Pv$ and $C_p = (\frac{\partial h}{\partial T})_P = C_V$ for incompressible fluid
- Transport properties: $\vec{q} = -k\nabla T$ and $\vec{\vec{\tau}} = -\mu \nabla \vec{V}$ for constant properties

And in cartesian coordinates:

$$\rho . C_p (\frac{\partial T}{\partial t} + V_x \frac{\partial T}{\partial x} + V_y \frac{\partial T}{\partial y} + V_z \frac{\partial T}{\partial z}) = k.(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}) + \mu . \phi_v$$

A similar derivation can be performed in cylindrical and spherical coordinates.

TABLE 10.2–3

THE EQUATION OF ENERGY IN TERMS OF THE TRANSPORT PROPERTIES

(for Newtonian fluids of constant ρ and k)

(Eq. 10.1–25 with viscous dissipation terms included)

Rectangular coordinates:

$$\rho \hat{C}_p\left(\frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z}\right) = k\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right]$$
$$+ 2\mu\left\{\left(\frac{\partial v_x}{\partial x}\right)^2 + \left(\frac{\partial v_y}{\partial y}\right)^2 + \left(\frac{\partial v_z}{\partial z}\right)^2\right\} + \mu\left\{\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)^2\right.$$
$$\left. + \left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right)^2 + \left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right)^2\right\} \qquad (A)$$

Cylindrical coordinates:

$$\rho \hat{C}_p\left(\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r}\frac{\partial T}{\partial \theta} + v_z \frac{\partial T}{\partial z}\right) = k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2}\right]$$
$$+ 2\mu\left\{\left(\frac{\partial v_r}{\partial r}\right)^2 + \left[\frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta} + v_r\right)\right]^2 + \left(\frac{\partial v_z}{\partial z}\right)^2\right\} + \mu\left\{\left(\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right)^2\right.$$
$$\left. + \left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right)^2 + \left[\frac{1}{r}\frac{\partial v_r}{\partial \theta} + r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)\right]^2\right\} \qquad (B)$$

Spherical coordinates:

$$\rho \hat{C}_p\left(\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r}\frac{\partial T}{\partial \theta} + \frac{v_\phi}{r \sin\theta}\frac{\partial T}{\partial \phi}\right) = k\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)\right.$$
$$\left. + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}\right] + 2\mu\left\{\left(\frac{\partial v_r}{\partial r}\right)^2\right.$$
$$\left. + \left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)^2 + \left(\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta \cot\theta}{r}\right)^2\right\}$$
$$+ \mu\left\{\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]^2 + \left[\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\right]^2\right.$$
$$\left. + \left[\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{v_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}\right]^2\right\} \qquad (C)$$

Note: The terms contained in braces { } are associated with viscous dissipation and may usually be neglected, except for systems with large velocity gradients.