Measurement, Design, and Analytic Techniques in Mental Health and Behavioral Sciences

Lecture 8 (April 23, 2009): Causal Inferences using Rubin's model

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Rubin’s causal model

- Measuring effects of causes or treatments. Effect of a cause is always relative to another cause. For example, "A causes B" means that A causes B relative to some other cause that includes the condition "not A".
- It is critical that each unit can be potentially (regardless of whether it can be achieved in practice) exposable to any one of the causes.
- As an example, the schooling a student receives can be a cause of a student’s performance on a test, whereas the student’s race or gender cannot.

Rubin's causal model

- Rubin's model is a quadruple $R = (U, K, Y, S)$
- U: the population of units
- K: a set of various causes or treatments under consideration
- Y is a real-valued function (response) defined on UxK, and $Y_t(u) = Y(u, t)$ is the value of the response that would be measured on u if u were exposed to cause t.
- S is a mapping from U to K, indicating the cause to which each unit u in U is exposed.

Definition of Causal Effects

- Treatment t causes the effect $Y_t(u) - Y_c(u)$ on unit u (relative to treatment c).
- The average causal effect, T, of t (relative to c) over U is

$$T = E(Y_t(u) - Y_c(u)).$$

What can be causes?

- Holland takes a position that causes are only those things that could, in principle, be treatments and can be manipulated in experiment.
- The notion of cause that operates in an experiment and in observational study is the same.
- The difference is in the degree of control over the phenomena under investigation.
- An attribute cannot be a cause in an experiment, because the notation of potential exposability does not apply to it.

Three Examples

- (a) She did well on the exam because she is a woman.
- (b). She did well on the exam because she studies for it.
- (c) She did well on the exam because she was coached by her teacher.
- The effect in the three statements is the same, doing well on an exam.
- In (A), the "cause" is ascribed to an attribute she possesses. In (B) the "cause" is ascribed to some voluntary activity she performed. In (C) the "cause" is the activity imposed on her.

Three examples, cont

- An attribute cannot be a cause because the notion of potential exposability does not apply to it.
- The only way for an attribute to change its value is for the unit to change in some way so that it’s no longer the same unit.
- Statements of "causation" involving attributes are always statements of association between the values of an attribute and a response variable across the units in a population.

Three examples, cont

- In (A) it meant that the performance of women on the exam exceeds that of men.
- Example (C) can be easily interpreted in terms of the causal model. The interpretation is that had she not been coached by her teacher she would not have done as well as she did.

Three examples, cont

- Example (B) illustrates a case where the applicability of the causal model is not absolutely clear because of the voluntary aspect of the supposed cause - studying for the exam.
- It is not clear that we could expose a person to studying or not in any verifiable way.
- We could operationally define studying as many hours of "nose in book", but that just define an attribute we could measure on a subject.
- Voluntary nature of much of human activity makes causal statements about these activities difficult in many cases.

Applicability of Causal Model

- To apply Rubin's model, it is important o decide when something is an attribute of units and when it is a cause that can act on units.
- The former case all that can be said is association, whereas in the latter case it may be possible to discuss measuring causal effects.

Randomized Trials with Non-compliance

- Z_i= the randomized treatment assignment indicator for the ith patient. $\mathbf{Z} = (Z_1, \ldots, Z_N)'$.
- $D_i(\mathbf{Z})$=the indicator for whether the ith patient would take the treatment given the vector of assignments $\mathbf{Z}$ for the N patients. $\mathbf{D} = (D_1(\mathbf{Z}), \ldots, D_N(\mathbf{Z}))'$.
- $Y_i(\mathbf{Z}, \mathbf{D})$: the outcome of the ith patient given the vector of treatments received $\mathbf{D}$ and the vector of treatment assignments $\mathbf{Z}$.

Potential outcomes

- Both $D_i(\mathbf{Z})$ and $Y_i(\mathbf{Z}, \mathbf{D})$ are referred as “potential outcomes”.

Stable Unit Treatment Value (SUTVA)

- Potential outcomes for each particular unit don't depend on the treatment status of other unit.
- This assumption disallows interference between units.
- Formally, the SUTVA assumes that

$$D_i(\mathbf{Z}) = D_i(\mathbf{Z}') \text{ and } Y_i(\mathbf{Z}, \mathbf{D}) = Y_i(\mathbf{Z}', \mathbf{D}') \text{ if } \mathbf{Z} = \mathbf{Z}'.$$

- Under the SUTVA assumption, we can write

$$D_i(\mathbf{Z}) = D_i(Z_i), Y_i(\mathbf{Z}, \mathbf{D}) = Y_i(Z_i, D_i).$$

Potential outcomes

- The potential outcomes of the ith patient: $(D_i(1), D_i(0))$, $(Y_i(1, D_i(1)), Y_i(0, D_i(0)))$.
- $Y_i(1, D_i(1))$: the response of the ith patient if is assigned to the treatment.
- $Y_i(0, D_i(0))$: the response of the ith patient if this patient's is assigned to the control.

Fundamental problem of causal analysis

- Either $(Y_i(1, D_i(1)), D_i(1))$ or $(Y_i(0, D_i(0)), D_i(0))$ is observed. But, we cannot observe both pairs.
- Another critical feature is that although the treatment assignment groups are random, treatment actually received groups are not random.

Definition of causal effects Z on Y

- Causal effect of Z on Y of the ith patient is

$$Y_i(1, D_i(1)) - Y_i(0, D_i(0)).$$

- The average causal effect of Z on Y is

$$ITT = E(Y_i(1, D_i(1)) - Y_i(0, D_i(0))).$$

Causal effects of D on Y

- To define causal effects of D on Y in a meaningful way we have to impose more restrictions on the potential outcomes.
- For example, we can make exclusion restriction: $Y_i(z, d) = Y_i(z', d')$ as long as $d = d'$.
- Then, $Y_i(z, D_i(z)) = Y_i(D_i(z))$.
- The causal effect of D on Y for subject i,

$$Y_i(1) - Y_i(0).$$

- The average causal effect of D on Y is

$$E(Y_i(1) - Y_i(0)).$$

Causal effects of D on Y, cont

- Without additional assumptions, such as exclusion restriction, we may not be able to define a global causal effect of D on Y, we can define the local causal effects to be discussed later.

Compliance Types

- Partition the population of patients into four sub-populations according to their compliance behavior.
- For the ith patient, he/she can be one of four types: complier, never-taker, always-taker, defier.
- Denote the compliance behavior of the ith patient by C_i:

$$C_i = \begin{cases} c \text{ i.e., the ith patient is a complier} & \text{if } D_i(z) = z, z = 0, 1 \\ n \text{ i.e., the ith patient is a never taker} & \text{if } D_i(z) = 0, z = 0, 1 \\ a \text{ i.e., the ith patient is always-taker} & \text{if } D_i(z) = 1, z = 0, 1 \\ d \text{ i.e., the ith patient is a defier} & \text{if } D_i(z) = 1 - z, z = 0, 1 \end{cases}$$

Local causal effects Z on Y

- Note that we have

$$ITT = E(Y_i(1, D_i(1)) - Y_i(0, D_i(0))) = \sum_{t=c,n,a,d} ITT_t P(C_i = t),$$

where

$$ITT_t = E(Y_i(1, D_i(1)) - Y_i(0, D_i(0)) \mid C_i = t).$$

- ITT_n: $Y_i(1, D_i(1)) = Y_i(1, 0)$ and $Y_i(0, D_i(0)) = Y_i(0, 0)$.
- ITT_a: $Y_i(1, D_i(1)) = Y_i(1, 1)$ and $Y_i(0, D_i(0)) = Y_i(0, 1)$.
- We call ITT_c as Complier Average Causal Effect (CACE).
- We call ITT_d as defier Average Causal Effect (DACE).

Estimation with Instrumental Variable (IV)

- Exclusion restriction
- Monotonicity assumption: $D_i(1) \geq D_i(0)$.
- The IV method can be used to estimate $CACE$.
- With monotonicity assumption, defiers do not exist.

Estimation with Instrumental Variable (IV), cont

- Under exclusion restriction, we have

$$Y_i(1, D_i(1)) - Y_i(0, D_i(0)) = Y_i(D_i(1)) - Y_i(D_i(0))$$

$$= [Y_i(1)D_i(1)+Y_i(0)(1-D_i(1))]-[Y_i(0)D_i(0)+Y_i(0)(1-D_i(0))]$$

$$= [Y_i(1) - Y_i(0)].[D_i(1) - D_i(0)]$$

- Under monotonicity assumption, $C_i = c, n, a$. Hence,

$$E[Y_i(1, D_i(1)) - Y_i(0, D_i(0)) = Y_i(D_i(1)) - Y_i(D_i(0))]$$

$$= E[Y_i(1) - Y_i(0)].[D_i(1) - D_i(0)]$$

$$= E[Y_i(1) - Y_i(0) \mid D_i(1) - D_i(0) = 1].P(D_i(1) - D_i(0) = 1).$$

Estimation with Instrumental Variable (IV), cont

- Since $D_i(1) - D_i(0) = 1$ is equivalent to $C_i = c$, we have

$$E[Y_i(1, D_i(1)) - Y_i(0, D_i(0)) = E[Y_i(1) - Y_i(0) \mid C_i = c].P(D_i(1) - D_i(0) = 1).$$

- Hence,

$$CACE = \frac{E[Y_i(1, D_i(1)) - Y_i(0, D_i(0))]}{P(D_i(1) - D_i(0) = 1)}.$$

- Because of randomization on Z, we have that

$$P(D_i(1) - D_i(0) = 1) = E[D_i(1) - D_i(0)] = E[D_i(1)] - E[D_i(0)]$$

$$= E[D_i(1) \mid Z_i = 1] - E[D_i(0) \mid Z_i = 0]$$

and that

$$E[Y_i(1, D_i(1)) - Y_i(0, D_i(0))] = E[Y_i(1, D_i(1)) \mid Z_i = 1] - E[Y_i(0, D_i(0)) \mid Z_i = 0].$$

Estimation with Instrumental Variable (IV), cont

- The IV estimator for $CACE$ is given by

$$\frac{\sum_{i=1}^{n} Y_i(1, D_i(1))Z_i / \sum_{i=1}^{n} Z_i - \sum_{i=1}^{n} Y_i(0, D_i(0))(1 - Z_i) / \sum_{i=1}^{n}(1 - Z_i)}{\sum_{i=1}^{n} D_i(1)Z_i / \sum_{i=1}^{n} Z_i - \sum_{i=1}^{n}(1 - D_i(0))(1 - Z_i) / \sum_{i=1}^{n}(1 - Z_i)}.$$

Notation

- For subject i, observed data - Z_i, $D_i = D_i(Z_i)$, and $Y_i = Y_i(Z_i, D_i)$.
- Missing-data - $D_i(1 - Z_i)$ and $Y_i(1 - Z_i, D_i(1 - Z_i))$.
- Let $S(z, d)$ be the subset of patients with $Z_i = z$ and $D_i = d$.
- Let N_{zd} be the number of elements in $S(z, d)$ and r_{zd} be # of $Y_i = 1$ in $S(z, d)$.
- That is, the observed data:

$$N_{zd} = \sum_{i=1}^{N} I_{[Z_i=z, D_i=d]}, r_{zd} = \sum_{i=1}^{N} Y_i I_{[Z_i=z, D_i=d]}.$$

- Parameters

$$\eta_{zt} = P(Y_i(z, D_i(z)) = 1 \mid Z_i = z, C_i = t), \omega_t = P(C_i = t), \xi_z = P(Z_i = z),$$

where $z = 0, 1$ and $t = n, a, c, d$.

Issues of identifiability

- Degree of freedom in the observed data (forming a contingency table ($D \times Z \times Y$): $8 - 1 = 7$.
- Without monotonicity and exclusion restriction, # of parameters: 8+3+1=12.
- There are 5 parameters are not estimable from the data

Role of assumptions

- Under the monotonicity assumption, we have $\omega_d = 0$.
- In addition,
  - for $S(0,0)$ ($D_i(0) = 0$), patients can have either never-takers or compliers ($C_i = n$ or c).
  - For $S(1,0)$ ($D_i(1) = 0$), patients are never-takers ($C_i = n$)
  - For $S(1,1)$ ($D_i(1) = 1$), patients are always-takers or compliers ($C_i = a$ or c).
  - For $S(0,1)$, patients are always-takers ($C_i = a$).

Role of assumptions, cont

- Under the exclusion restriction assumption on all t, we have $\eta_{zt} = \eta_t$, $t = n, a, c$.
- Hence under both the monotonicity and exclusion restriction assumptions, the number of parameters is 3 + 2 +1=6.
- We only need to make the exclusion restriction assumption for $t = n$ and $t = a$. That is, $\eta_{zn} = \eta_n$ and $\eta_{za} = \eta_a$. Then, the number of parameters is 4+2+1=7.

Moment methods

- We first derive moment estimators for η_n and η_a.
- Note that the set that $Z_i = 1$ and $C_i = n$ is equivalent to the set that $Z_i = 1$ and $D_i = 0$ because of monotonicity assumption.
- Hence, we have

$$\eta_n = P(Y_i = 1 \mid Z_i = 1, C_i = n) = \frac{P(Y_i = 1, Z_i = 1, C_i = n)}{P(Z_i = 1, C_i = n)} =$$

$$\frac{P(Y_i = 1, Z_i = 1, D_i = 0, C_i = n)}{P(Z_i = 1, D_i = 0, C_i = n)} = \frac{P(Y_i = 1, Z_i = 1, D_i = 0)}{P(Z_i = 1, D_i = 0)}.$$

- Therefore, the moment estimators for η_n is

$$\widehat{\eta_n} = \frac{\sum_{i=1}^N Y_i Z_i (1 - D_i)}{\sum_{i=1}^N Z_i (1 - D_i)} = \frac{r_{10}}{N_{10}}.$$

- Similarly, we have

$$\eta_a = \frac{P(Y_i = 1, Z_i = 0, D_i = 1)}{P(Z_i = 0, D_i = 1)}.$$

Moment Estimators, cont

- Hence, the moment estimator for η_a is given by

$$\widehat{\eta}_a = \frac{\sum_{i=1}^{N} Y_i(1-Z_i)D_i}{\sum_{i=1}^{N}(1-Z_i)D_i} = \frac{r_{01}}{N_{01}}.$$

Moment estimators for ω_t

- We next derive moment estimators for ω_t.
- Because of randomization, we have that

$$\omega_n = P(C_i = n \mid Z_i = 1) = \frac{P(C_i = n, Z_i = 1, D_i = 0)}{P(Z_i = 1)} = \frac{P(Z_i = 1, D_i = 0)}{P(Z_i = 1)}.$$

- Hence, the moment estimator for ω_n is given by

$$\widehat{\omega}_n = \frac{\sum_{i=1}^N Z_i(1 - D_i)}{\sum_{i=1}^N Z_i}.$$

- Similarly, we obtain the following moment estimator for ω_a:

$$\widehat{\omega}_a = \frac{\sum_{i=1}^N (1 - Z_i)D_i}{\sum_{i=1}^N (1 - Z_i)}.$$

- The moment estimator for ω_c is given by

$$\widehat{\omega}_c = 1 - \widehat{\omega}_n - \widehat{\omega}_a.$$

Moment estimation - cont

- Next we derive moment estimators for outcome parameters in complier-type, η_{0c} and η_{1c}.
- Since $(Z_i = 0, D_i = 0) \equiv (Z_i = 0, D_i = 0, C_i = c) \cup (Z_i = 0, D_i = 0, C_i = n)$, we have that

$$P(Y_i = 1 \mid Z_i = 0, D_i = 0) =$$

$$P(Y_i = 1 \mid Z_i = 0, D_i = 0, C_i = n)P(C_i = n \mid Z_i = 0, D_i = 0) +$$

$$P(Y_i = 1 \mid Z_i = 0, D_i = 0, C_i = c)P(C_i = c \mid Z_i = 0, D_i = 0).$$

Note that

$$P(C_i = c \mid Z_i = 0, D_i = 0) = \frac{P(C_i = c, Z_i = 0, D_i = 0)}{\sum_{t=n,c} P(Z_i = 0, D_i = 0, C_i = t)} =$$

$$\frac{P(D_i = 0 \mid C_i = c, Z_i = 0)P(C_i = c)P(Z_i = 0)}{\sum_{t=n,c} P(D_i = 0 \mid Z_i = 0, C_i = t)P(C_i = t)P(Z_i = 0)} = \frac{\omega_c}{\omega_c + \omega_n}.$$

Moment Estimator, Cont

- And,

$$P(C_i = n \mid Z_i = 0, D_i = 0) = \frac{\omega_n}{\omega_c + \omega_n}.$$

- Hence

$$P(Y_i = 1 | Z_i = 0, D_i = 0) = \eta_n \frac{\omega_n}{\omega_c + \omega_n} + \eta_{0c} \frac{\omega_c}{\omega_c + \omega_n}.$$

- Similarly, we can show that

$$P(Y_i = 1 | Z_i = 1, D_i = 1) = \eta_a \frac{\omega_a}{\omega_c + \omega_a} + \eta_{1c} \frac{\omega_c}{\omega_c + \omega_a}.$$

Moment estimation - cont

- Hence by solving the above two equations for η_{0c} and η_{1c}, we obtain the moment estimators for η_{oc} and η_{1c} as follows:

$$\widehat{\eta}_{0c} = [\frac{\sum_{i=1}^{N} Y_i(1-Z_i)(1-D_i)}{\sum_{i=1}^{N}(1-Z_i)(1-D_i)} - \widehat{\eta}_n \frac{\widehat{\omega}_n}{\widehat{\omega}_c + \widehat{\omega}_n}] \frac{\widehat{\omega}_c + \widehat{\omega}_n}{\widehat{\omega}_c},$$

and

$$\widehat{\eta}_{1c} = [\frac{\sum_{i=1}^{N} Y_i Z_i D_i}{\sum_{i=1}^{N} Z_i D_i} - \widehat{\eta}_a \frac{\widehat{\omega}_a}{\widehat{\omega}_c + \widehat{\omega}_a}] \frac{\widehat{\omega}_c + \widehat{\omega}_a}{\widehat{\omega}_c}.$$

Sommer-Zeger vitamin supplement data

Type	Assignment $Z_{obs,i}$	Vitamin supplements $D_{obs,i}$	Survival $Y_{obs,i}$	Number of units (Total 23,682)
Complier or never-taker	0	0	0	74
Complier or never-taker	0	0	1	11,541
Never-taker	1	0	0	34
Never-taker	1	0	1	2,385
Complier	1	1	0	12
Complier	1	1	1	9,663

Results

- The estimate under the exclusion for CACE is 0.0032 with the 90% confidence interval of $(0.0012, 0.0051)$.