Introduction to Structural Equation Models

Chuan Zhou, PhD.

Department of Pediatrics, University of Washington

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Outline

- Motivating example
- Specification of Structural Equation Models
- Instrumental variables estimation
- Identification problem
- Estimation of observed-variable SEMs
- General structural-equation models

Klein's macroeconomic model

Klein, L. 1950. *Economic Fluctuations in the United States 1921-1941*. New York, Wiley.

- Consumption = f(Private profits, Private wages, Government wages)
- Investment = f(Private profits, Capital stock)
- Private wages = f(Time trend, Spending Demand)

Klein's model in equations

Klein's economic model can be expressed in the following set of regression models,

$$
\begin{aligned}
C_t &= \gamma_{10} + \gamma_{11} P_t + \gamma_{12} P_{t-1} + \beta_{11}(W_t^p + W_t^g) + \zeta_{1t} \\
I_t &= \gamma_{20} + \gamma_{21} P_t + \gamma_{22} P_{t-1} + \beta_{21} K_{t-1} + \zeta_{2t} \\
W_t^p &= \gamma_{30} + \gamma_{31} A_t + \beta_{31} X_t + \beta_{32} X_{t-1} + \zeta_{3t} \\
X_t &= C_t + I_t + G_t \\
P_t &= X_t - T_t - W_t^p \\
K_t &= K_{t-1} + I_t
\end{aligned}
$$

C_t	Consumption (in year t)
I_t	Investment
W_t^p	Private wages
X_t	Equilibrium demand
P_t	Private profits
K_t	Capital stock
G_t	Government non-wager spending
T_t	Indirect business taxes and net exports
W_t^g	Government wages
A_t	Time trend, year-1831

Structural Equation Models

- *Structural-equation models (SEMs)* are multiple-equation regression models in which the response variable in one regression equation can appear as an explanatory variable in another equation
- Structural-equation models can include variables that are not measured directly, but rather indirectly through their effects (*indicators*) or, sometimes, through observed causes (menifest variables)
- Model structural-equation methods represent a confluence of work in many disciplines, including biostatistics, econometrics, psychometrics, etc.

Steps of SEM

- Specify the model (has to be *a priori*)
- Determine whether the model if identified
- Select measures of the variables and collect the data
- Analyze the model
- Evaluate model fit
- Respecify the model

Some cautionary notes

- SEMs are multiple-equation regression models representing putative causal (and hence *structural*) relationships among a number of variables, some of which may affect one another mutually.
- Design is rarely explicitly taken into account, mostly on observational data
- Lack of sound conceptual framework for causal effects
- Claiming that a relationship is causal based on observational data is intrinsically problematic and requires support beyond the data at hand

Two classes of variables

- *Endogenous variables* are the response variables of the model
  - In path diagram, they are the nodes with directional arrows going into
  - One structural equation per endogenous variable
  - An endogenous variable may also be an explanatory variable in other structural equations
- *Exogenous variables* appear only as explanatory variables in the SEMs
  - In path diagram, they are the nodes without arrows going into
  - the values of exogenous variables are therefore determined outside of the model
  - Assumed to be measured without error (unless latent)
  - Can be categorical while endogenous variables are mostly continuous

Structural errors

- Aaggregated omitted causes of the endogenous variables plus measurement error (and possibly intrinsic randomness) in the endogenous variables
- One error variable per endogenous variable
- Assumed to have zero expectation and to be independent of exogenous variables
- Errors for different observations are assumed to be independent, but maybe correlated within observation
- Each error variable is assumed to have constant variance across observations, although the variances may differ across error variables
- Sometimes normality is assumed

Structural coefficients and covariance

- Structural coefficients represent the direct (partial) effect
  - on directed edge in path diagram
  - of an exogenous on an endogenous variable
  - of an endogenous on another endogenous variable
- Covariances can be either between two exogenous variables or two error variables (unanalyzed associations)

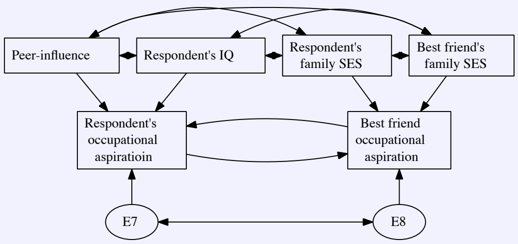
Path diagrams

Path diagram is a causal graph commonly used in SEMs. Some conventions are

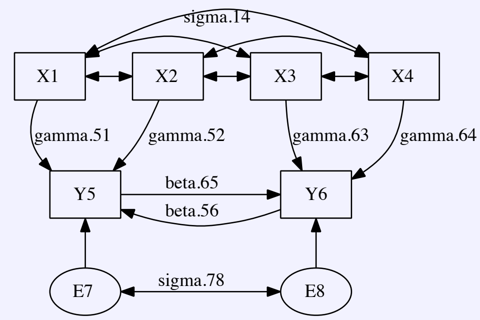
- Nodes: observed variables in boxes, latent variables in circles
- Edges: a directed (single headed) arrow represent a direct effect of one variable on another; a bidirectional arrow represents a covariance (no causal interpretation given)
- Labels: unique subscripts on variables are helpful

A path diagram example

Duncan, Haller, and Portes's (1968) study of peer influence on the aspiration of high school students.



Simplify the labels



Structural equations

- The structural equations of a model can be read straightforwardly from the path diagram.

$$y_5 = \gamma_{51}x_1 + \gamma_{52}x_2 + \beta_{56}y_6 + \epsilon_7$$
$$y_6 = \gamma_{63}x_3 + \gamma_{64}x_4 + \beta_{65}y_5 + \epsilon_8$$

- With some manipulation, including centering the exogenous variables at the means

$$\begin{bmatrix} 1 & -\beta_{56} \\ -\beta_{65} & 1 \end{bmatrix} \begin{bmatrix} y_5 \\ y_6 \end{bmatrix} + \begin{bmatrix} -\gamma_{51} & -\gamma_{52} & 0 & 0 \\ 0 & 0 & -\gamma_{63} & -\gamma_{64} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \epsilon_7 \\ \epsilon_8 \end{bmatrix}$$

Matrix form of the model

- More generally, when there are q endogenous variables, q errors, and m exogenous variables, the model for an individual observation is

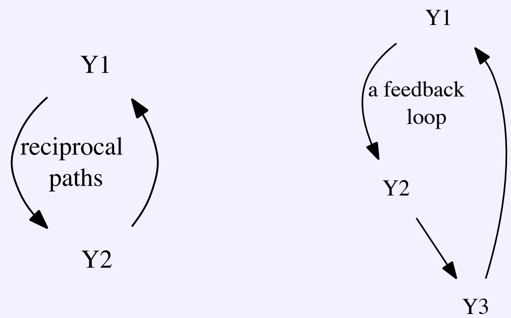
$$\underset{(q\times q)}{\mathbf{B}}\underset{(q\times 1)}{\mathbf{y}_i} + \underset{(q\times m)}{\mathbf{\Gamma}}\underset{(m\times 1)}{\mathbf{x}_i} = \underset{(q\times 1)}{\epsilon_i}$$

- For all n observations,

$$\underset{(n\times q)}{\mathbf{Y}}\underset{(q\times q)}{\mathbf{B}'} + \underset{(n\times m)}{\mathbf{X}}\underset{(m\times q)}{\mathbf{\Gamma}'} = \underset{(q\times 1)}{\mathbf{E}}$$

Recursive models

- An important type of SEM, called a *recursive* model, has two defining characteristics:
  1. Different error variables are independent
  2. There are no reciprocal directed paths or feedback loops in the path diagram
- Put another way, the error covariance matrix $\boldsymbol{\Sigma}_{\epsilon\epsilon}$ is diagonal, while $\mathbf{B}$ matrix is lower-triangular



Estimation for recursive models

- As a consequence of the two properties of recursive models, the predictors are always independent of the error, and the model can be estimated by a sequence of OLS regressions
- SEMs that are not recursive are termed *nonrecursive*
- There are also *block resursive* SEMs

Instrumental Variables

- Instrumental-variable (IV) estimation serves two purposes: check whether the model is identifiable and estimate the structural coefficients if it is
- An *instrument variable* is a variable uncorrelated with the error of a structural equation AND correlated with an exogenous variable

Simple regression

- To understand the IV approach to estimation, consider the following simple linear regression

$$y = \beta x + \epsilon$$

where $\mathrm{E}(\epsilon) = 0$, $\mathrm{var}(\epsilon) = \sigma_\epsilon^2$, x and ϵ are independent.
- Now multiply both sides of the model by x and take expectations,

$$\begin{aligned} \mathrm{cov}(x, y) &= \beta \mathrm{var}(x) + \mathrm{cov}(x, \epsilon) \\ \sigma_{xy} &= \beta \sigma_x^2 + 0 \end{aligned}$$

- Plug in consistent sample estimates and solve for β

$$b = \frac{s_{xy}}{s_x^2} = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sum(x_i - \bar{x})^2}$$

IV with simple regression

- Imagine, alternatively, that x and ϵ are not independent, but ϵ is independent of some other variable z
- Suppose further that z and x are correlated, that is, $\mathrm{cov}(z, x) \neq 0$
- Then, proceed as before but with z,

$$\begin{aligned} \mathrm{cov}(z, y) &= \beta \mathrm{cov}(z, x) + \mathrm{cov}(z, \epsilon) \\ \sigma_{zy} &= \beta \sigma_{zx} + 0 \\ \beta &= \frac{\sigma_{zy}}{\sigma_{zx}} \end{aligned}$$

- $b_{IV} = \frac{s_{zy}}{s_{zx}} = \frac{\sum(z_i - \bar{z})(y_i - \bar{y})}{\sum(z_i - \bar{z})(x_i - \bar{x})}$

Instrumental-variable estimation in matrix form

- Now consider

$$\underset{(n\times 1)}{\mathbf{y}} = \underset{(n\times(k+1))}{\mathbf{X}} \underset{(k+1)\times 1}{\boldsymbol{\beta}} + \underset{(n\times 1)}{\boldsymbol{\epsilon}},$$

where $\boldsymbol{\epsilon} \sim \mathrm{N}_n(\mathbf{0}, \sigma_\epsilon^2 \mathbf{I}_n)$.
- When $\mathbf{X}$ and $\boldsymbol{\epsilon}$ are independent, $\mathbf{b}_{OLS} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$
- When $\mathbf{X}$ and $\boldsymbol{\epsilon}$ are NOT independent, suppose we have observations on $(k+1)$ instrumental variables $\underset{n\times(k+1)}{\mathbf{Z}}$, that are independent of $\boldsymbol{\epsilon}$, then follow the scalar treatment,

$$\mathbf{b}_{IV} = (\mathbf{Z}'\mathbf{X})^{-1}\mathbf{Z}'\mathbf{y}$$

is a consistent estimator of $\boldsymbol{\beta}$

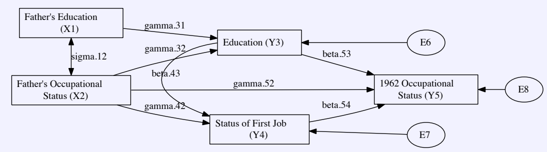
Identification problem

- SEM is *under-identified* if there are *fewer* instrumental variables than predictors
- SEM is *just-identified* if number of IVs is the same as predictors
- SEM is *over-identified* if there are *more* IVs than predictors, we can either discard surplus IVs, or use better method such as two-stage least squares
- For $\mathbf{b}_{IV}$ to be defined, in addition to at least $(k+1)$ IVs, we also need $\mathbf{Z}'\mathbf{X}$ to be non-singular
- It requires IVs are correlated with predictors plus there is no perfect collinearity

Estimation of recursive SEMs

- By its definition, pool of IVs for recursive SEMs contains exogenous variables and *prior* endogenous variables
- Always have at least as many IVs as predictors, therefore necessarily identified
- To understand this, consider Blau and Duncan's basic-stratification model, The American Occupational Structure (1967).

Blau and Duncan's basic-stratification model



Two-stage least squares (2SLS) estimation

- Using combination of IVs for estimation in *over-identified* non-recursive SEMs
- First stage, regress predictors $\mathbf{X}$ on the IVs $\mathbf{Z}$, obtaining fitted values
$$\hat{\mathbf{X}} = \mathbf{X}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{X}$$
- Second stage, the response $\mathbf{y}$ is regressed on $\hat{\mathbf{X}}$, producing the 2SLS estimator of β
$$\hat{\beta} = (\hat{\mathbf{X}}'\hat{\mathbf{X}})^{-1}\hat{\mathbf{X}}'\mathbf{y}$$
- Column of $\mathbf{X}$ are uncorrelated with the structural disturbance in the probability limit
- Very similar to weighted least squares!

Full information maximum likelihood (FIML) estimation

- Along with other standard assumptions of SEMs, FIML estimates are calculated under the assumption that the structural errors are multivariately normally distributed
- Under this assumption, the log-likelihood for the model is

$$\log_e L(\mathbf{B}, \mathbf{\Gamma}, \mathbf{\Sigma}_{\epsilon\epsilon}) = n \log \mid det(\mathbf{B}) \mid - \frac{nq}{2}\log 2\pi - \frac{n}{2}\log det(\mathbf{\Sigma}_{\epsilon\epsilon}) - \frac{1}{2}\sum_{i=1}^{n}(\mathbf{B}\mathbf{y}_i + \mathbf{\Gamma}\mathbf{x}_i)'\mathbf{\Sigma}_{\epsilon\epsilon}^{-1}(\mathbf{B}\mathbf{y}_i + \mathbf{\Gamma}\mathbf{x}_i)$$

- The full general machinery of MLE is available if the model is identifiable

Klein's model revisited

```
> library(sem)
> data(Klien)
> head(Klein)
  Year    C    P   Wp    I K.lag    X  Wg   G   T
1 1920 39.8 12.7 28.8  2.7 180.1 44.9 2.2 2.4 3.4
2 1921 41.9 12.4 25.5 -0.2 182.8 45.6 2.7 3.9 7.7
3 1922 45.0 16.9 29.3  1.9 182.6 50.1 2.9 3.2 3.9
4 1923 49.2 18.4 34.1  5.2 184.5 57.2 2.9 2.8 4.7
5 1924 50.6 19.4 33.9  3.0 189.7 57.1 3.1 3.5 3.8
6 1925 52.6 20.1 35.4  5.1 192.7 61.0 3.2 3.3 5.5
```


Klein's model revisited

```
> P.lag <- c(NA, P[-length(P)])
> X.lag <- c(NA, X[-length(X)])
> A <- Year -1931
> cbind(Year, A, P, P.lag, X, X.lag)
     Year   A    P P.lag    X X.lag
 [1,] 1920 -11 12.7    NA 44.9    NA
 [2,] 1921 -10 12.4  12.7 45.6  44.9
 [3,] 1922  -9 16.9  12.4 50.1  45.6
 [4,] 1923  -8 18.4  16.9 57.2  50.1
 [5,] 1924  -7 19.4  18.4 57.1  57.2
 [6,] 1925  -6 20.1  19.4 61.0  57.1
```


Klein's model revisited

```
> eqn.1 <- tsls(C~P+P.lag+I(Wp+Wg),
+ instruments=~G+T+Wg+A+P.lag+K.lag+X.lag, data=Klein)
> summary(eqn.1)

 2SLS Estimates
Model Formula: C ~ P + P.lag + I(Wp + Wg)
Instruments: ~G + T + Wg + A + P.lag + K.lag + X.lag
Residuals:
     Min.   1st Qu.    Median      Mean   3rd Qu.      Max.
-1.89e+00 -6.16e-01 -2.46e-01 -2.74e-12  8.85e-01  2.00e+00

            Estimate Std. Error t value  Pr(>|t|)
(Intercept) 16.55476    1.46798 11.2772 2.587e-09
P            0.01730    0.13120  0.1319 8.966e-01
P.lag        0.21623    0.11922  1.8137 8.741e-02
I(Wp + Wg)   0.81018    0.04474 18.1107 1.505e-12

Residual standard error: 1.1357 on 17 degrees of freedom
```


Duncan, Haller, and Portest peer influence model

```
> R.DHP <- read.moments(diag=FALSE, names=c('ROccAsp', 'REdAsp',
+ 'FOccAsp', 'FEdAsp', 'RParAsp', 'RIQ', 'RSES', 'FSES', 'FIQ',
+   'FParAsp'))
1:     .6247
2:     .3269  .3669
4:     .4216  .3275  .6404
7:     .2137  .2742  .1124  .0839
11:     .4105  .4043  .2903  .2598  .1839
16:     .3240  .4047  .3054  .2786  .0489  .2220
22:     .2930  .2407  .4105  .3607  .0186  .1861  .2707
29:     .2995  .2863  .5191  .5007  .0782  .3355  .2302  .2950
37:     .0760  .0702  .2784  .1988  .1147  .1021  .0931 -.0438  .2087
46:
Read 45 items
```

Duncan, Haller, and Portest peer influence model

```
> model.dhp <- specify.model()
1:     RParAsp  -> RGenAsp, gam11,  NA
2:     RIQ      -> RGenAsp, gam12,  NA
3:     RSES     -> RGenAsp, gam13,  NA
4:     FSES     -> RGenAsp, gam14,  NA
5:     RSES     -> FGenAsp, gam23,  NA
6:     FSES     -> FGenAsp, gam24,  NA
7:     FIQ      -> FGenAsp, gam25,  NA
8:     FParAsp  -> FGenAsp, gam26,  NA
9:     FGenAsp  -> RGenAsp, beta12, NA
10:     RGenAsp  -> FGenAsp, beta21, NA
11:     RGenAsp  -> ROccAsp,  NA,       1
12:     RGenAsp  -> REdAsp,  lam21,  NA
13:     FGenAsp  -> FOccAsp,  NA,       1
14:     FGenAsp  -> FEdAsp,  lam42,  NA
15:     RGenAsp <-> RGenAsp, ps11,   NA
...
```


Duncan, Haller, and Portest peer influence model

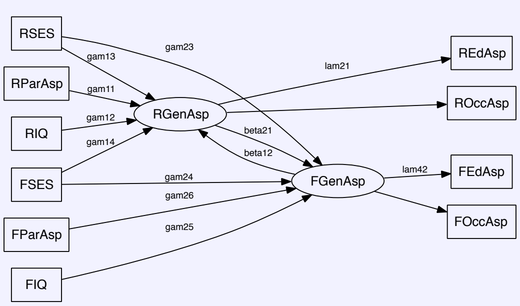
```
> sem.dhp <- sem(model.dhp, R.DHP, 329,
+     fixed.x=c('RParAsp', 'RIQ', 'RSES', 'FSES', 'FIQ', 'FParAsp'))
> summary(sem.dhp)

 Model Chisquare =  26.697   Df =  15 Pr(>Chisq) = 0.031302
 Chisquare (null model) =  872   Df =  45
 Goodness-of-fit index =  0.98439
 Adjusted goodness-of-fit index =  0.94275
 RMSEA index =  0.048759   90% CI: (0.014517, 0.07831)
 Bentler-Bonnett NFI =  0.96938
 Tucker-Lewis NNFI =  0.95757
 Bentler CFI =  0.98586
 SRMR =  0.020204
 BIC =  -60.244

 Parameter Estimates
       Estimate  Std Error z value Pr(>|z|)
gam11   0.161224 0.038487   4.1890 2.8019e-05 RGenAsp <--- RParAsp
...
```


A path diagram

The following path diagram was generated by path.diagram() function in sem package



General structural equation models

- Include unobserved exogenous or endogenous variables (also termed factors or latent variables) in addition to unobservable disturbances
- Sometimes called LISREL models (linear structural relations), after first widely available computer program (J oreskog, 1973)
- Mainly likelihood based estimation
- No simple general solution towards identification
- There are many ways to fool yourself with SEMs