Lecture 7: Eigenvalues & Eigenvectors

- Unlike the system $\mathbf{Ax} = \mathbf{b}$ which has the single unknown vector $\mathbf{x}$, eigenvalue problems are of the form $\mathbf{Ax} = \lambda\mathbf{x}$, with unknowns $\mathbf{x}$ and λ.

- The values of λ are known as the eigenvalues and the corresponding $\mathbf{x}$ are the eigenvectors.

- Eigenvalue problems often arise from differential equations.

- Consider the example of a linear set of coupled differential equations:

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} = \mathbf{A}\mathbf{y} \tag{1}$$

- Inserting the solution $\mathbf{y} = \mathbf{x}e^{(\lambda t)}$

- Results in: $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$

- Once the system is solved for all N linearly independent solution eigenvectors $\mathbf{x}$ and corresponding λ's for the $N \times N$ matrix $\mathbf{A}$

- The solution of the ODE is

  $\mathbf{y} = c_1\mathbf{x_1}e^{(\lambda_1 t)} + c_2\mathbf{x_2}e^{(\lambda_2 t)} + ... + c_N\mathbf{x_N}e^{(\lambda_N t)}$

- How to find these eigenvalues and eigen-vectors?

```
% Get the eigenvalues of a matrix A
% in MATLAB
eig(A)
help eig

% To obtain diagonal matrix D of
% eigenvalues, and a full matrix V
% whose columns are the corresponding
% eigenvectors
[V,D] = eig(A)
```

- Understanding eigenvalues and eigenvectors?

- Rewrite the eigenvalue problem $\mathbf{Ax} = \lambda\mathbf{x}$

  $\Rightarrow \mathbf{Ax} = \mathbf{I}\lambda\mathbf{x}$ where $\mathbf{I}$ is the identity matrix.

  $\Rightarrow \mathbf{Ax} - \mathbf{I}\lambda\mathbf{x} = 0$

  $\Rightarrow (\mathbf{A} - \mathbf{I}\lambda)\mathbf{x} = 0$

Now there are two possibilities for solutions:

- $\mathbf{I}$. $det(\mathbf{A} - \mathbf{I}\lambda) \neq \mathbf{0}$

  $\Rightarrow \mathbf{x} = (\mathbf{A} - \mathbf{I}\lambda)^{-1}\mathbf{0}$

  $\Rightarrow \mathbf{x} = \mathbf{0}$ which doesn't help us.

- $\mathbf{II}$. $det(\mathbf{A} - \mathbf{I}\lambda) = \mathbf{0}$

  $\Rightarrow (\mathbf{A} - \mathbf{I}\lambda)$ is singular

  $\Rightarrow (\mathbf{A} - \mathbf{I}\lambda)^{-1}$ does not exist

  $\Rightarrow$ we can find $\mathbf{x}$ so that $(\mathbf{A} - \mathbf{I}\lambda)\mathbf{x} = 0$

- To solve the eigenvalue problem we take two steps:

  1. Find λ such that $(\mathbf{A} - \mathbf{I}\lambda)\mathbf{x} = 0$

  2. Find $\mathbf{x}$ such that $(\mathbf{A} - \mathbf{I}\lambda)\mathbf{x} = 0$

- Solve an example eigenvalue problem by hand (first rewriting the problem in matrix notation) given the following ordinary differential equations

$$\begin{aligned}\frac{\mathrm{d}y_1}{\mathrm{d}t} &= 2y_1 + 4y_2 \\ \frac{\mathrm{d}y_2}{\mathrm{d}t} &= 3y_1 + y_2\end{aligned}$$

- Rewriting the ODE system as a matrix system

$$\begin{pmatrix} \frac{\mathrm{d}y_1}{\mathrm{d}t} \\ \frac{\mathrm{d}y_2}{\mathrm{d}t} \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{2}$$

- Rewrite matrix system as eigenvalue problem

$$(\mathbf{A} - \lambda\mathbf{I}) = \begin{pmatrix} 2-\lambda & 4 \\ 3 & 1-\lambda \end{pmatrix} \tag{3}$$

- Step 1 and step 2 **by hand lol**

- Note that eigenvectors can be scaled, ie. if $\mathbf{x}$ is an eigenvector then $c\mathbf{x}$ is also an eigenvector, where c is scalar. These two eigenvectors are linearly dependent.

- MATLAB returns eigenvectors in a scaling that results in normalized vectors, that is $\|\mathbf{x}\| = 1$

```
% Try the same eigenvalue problem in MATLAB
[V,D] = eig([2 4; 3 1])
norm(V(:,1))
```

- Computing $\mathbf{A}^M$ for large M using direct matrix multiplication is expensive: $mO(N^3)$ for an N by N matrix $\mathbf{A}$.

- Alternative using the eigenvalues and eigenvectors, assuming we have all n eigenvalues and eigenvectors

$$\begin{aligned}
\mathbf{Ax_1} &= \lambda_1\mathbf{x_1} \\
\mathbf{Ax_2} &= \lambda_2\mathbf{x_2} \\
&. \\
&. \\
&. \\
\mathbf{Ax_n} &= \lambda_n\mathbf{x_n}
\end{aligned}$$

- Put all the eigenvectors in a matrix $\mathbf{S} = (\mathbf{x_1}, \mathbf{x_2}, ..., \mathbf{x_n})$ and all n eigenvalues in the diagonal of an n by n matrix $\mathbf{\Lambda}$ then

  $\mathbf{AS} = \mathbf{S\Lambda}$

  $\Rightarrow$

$$\begin{aligned}
\mathbf{A^1} &= \mathbf{S\Lambda S^{-1}} \\
\mathbf{A^2} &= (\mathbf{S\Lambda S^{-1}})(\mathbf{S\Lambda S^{-1}}) = (\mathbf{S\Lambda^2 S^{-1}}) \\
\mathbf{A^3} &= (\mathbf{S\Lambda^2 S^{-1}})(\mathbf{S\Lambda S^{-1}}) = (\mathbf{S\Lambda^3 S^{-1}}) \\
&\ . \\
&\ . \\
&\ . \\
\mathbf{A^M} &= (\mathbf{S\Lambda^{M-1} S^{-1}})(\mathbf{S\Lambda S^{-1}}) = (\mathbf{S\Lambda^M S^{-1}})
\end{aligned}$$

- Since $\mathbf{\Lambda}$ is a diagonal matrix, we only need to take the power of the diagonal elements to get the power of the entire matrix.