Lecture 17: BVP Direct and Relaxation

Consider

$$\frac{d^2y(t)}{dt^2} = p(t)\frac{dy(t)}{dt} + q(t)y(t) + r(t)$$

on $t \in [a,b]$ with boundary conditions

$$\begin{aligned} y(a) &= \alpha \\ y(b) &= \beta \end{aligned}$$

Using the second order central differences:

$$\begin{aligned} f'(t) &= \frac{f(t+\Delta t) - f(t-\Delta t)}{2\Delta t} \\ f''(t) &= \frac{f(t+\Delta t) - 2f(t) + f(t-\Delta t)}{\Delta t^2} \end{aligned}$$

$$\frac{y(t+\Delta t)-2y(t)+y(t-\Delta t)}{\Delta t^2}$$
$$= p(t)\frac{y(t+\Delta t)-y(t-\Delta t)}{2\Delta t}+q(t)y(t)+r(t)$$

This can be re-written as:

$A(t)y(t-\Delta t)+B(t)y(t)+C(t)y(t+\Delta t)=rhs(t)$

where

$$\begin{aligned}
A(t) &= [1+\frac{\Delta t}{2}p(t)] \\
B(t) &= -[2+\Delta t^2 q(t)] \\
C(t) &= [1-\frac{\Delta t}{2}p(t)] \\
rhs(t) &= [\Delta t^2 r(t)]
\end{aligned}$$

The boundary conditions are implemented as:

$$\begin{aligned}
y(t_0) &= y(a)=\alpha \\
y(t_N) &= y(b)=\beta
\end{aligned}$$

The system of equations can then be written as a matrix problem $\mathbf{Ax} = \mathbf{b}$ where

$$\begin{pmatrix} B(t_1) & C(t_1) & 0 & ... & & 0 \\ A(t_2) & B(t_2) & C(t_2) & ... & & 0 \\ 0 & A(t_3) & B(t_3) & C(t_3) & ... & 0 \\ . & ... & & & & 0 \\ & & & & & \\ . & ... & & A(t_{N-2}) & B(t_{N-2}) & C(t_{N-2}) \\ 0 & ... & & 0 & A(t_{N-1}) & B(t_{N-1}) \end{pmatrix}$$

$$\mathbf{x} = [y(t_1), y(t_2), ..., y(t_{N-2}), y(t_{N-1})]'$$

$$\mathbf{b} = \begin{pmatrix} rhs(t_1) - A(t_1)y(t_0) \\ rhs(t_2) \\ rhs(t_3) \\ . \\ . \\ . \\ rhs(t_{N-2}) \\ rhs(t_{N-1}) - C(t_{N-1})y(t_N) \end{pmatrix}$$

The system $\mathbf{Ax} = \mathbf{b}$ can then be solved in the usual manner shown previously, either with a direct method or an iterative method.

We will now introduce the Newton-Raphson iterative method (alternatively known as Newton's method) which can be used for non-linear equations (which may or may not have a guaranteed or unique solution):

A non-linear case example could be where the function $q(t)$ above is not only a function of t but also say of $y(t)$ and $y'(t)$ such as

$$q(t, y(t), y'(t)) = ty'(t)$$

Newton's method For a function $f(x)$ where we need to find x which satisfies $f(x) = 0$ and where $f'(x) \neq 0$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

with an initial guess x_0. Begin by considering a non-linear equation to apply Newton's method

$$\begin{aligned} & cos(2\pi x) = 1 \\ \Rightarrow \quad & f(x) = cos(2\pi x) - 1 = 0 \end{aligned}$$

thus $f'(x) = -2\pi sin(2\pi x)$

Type in MATLAB Newton's method with initial condition $x0 = .7$:

```
x0 = .7
x1 = x0+1;
while (abs(x1-x0) > 10^-6)
  x0 = x1;
  x1 = x0 - (cos(2*pi*x0)-1)/(-2*pi*sin(2*pi*x0))
end
```

Generalizing Newton's method for a system of K non-linear equations given by

$$\mathbf{F}(\mathbf{x_n}) = \begin{pmatrix} f_1(\mathbf{x_n}) \\ f_2(\mathbf{x_n}) \\ . \\ . \\ . \\ f_{K-1}(\mathbf{x_n}) \\ f_K(\mathbf{x_n}) \end{pmatrix}$$

$$\begin{aligned} \mathbf{x}_{n+1} &= \mathbf{x}_n + \mathbf{\Delta x_n} \\ \mathbf{J}(\mathbf{x_n})\mathbf{\Delta x_n} &= -\mathbf{F}(\mathbf{x_n}) \end{aligned}$$

where $\mathbf{J}(\mathbf{x_n})$ is called the Jacobian matrix given by

$$\begin{pmatrix} \frac{df_1}{dx_1} & \frac{df_1}{dx_2} & \cdots & \frac{df_1}{dx_K} \\ \frac{df_2}{dx_1} & \frac{df_2}{dx_2} & \cdots & \frac{df_2}{dx_K} \\ . & & & \\ . & & & \\ . & & & \\ \frac{df_K}{dx_1} & \cdots & & \frac{df_K}{dx_K} \end{pmatrix}$$