# Lecture 12: Numerical Integration

See pictures of numerical integration courtesy of Eric Shea-Brown.

Numerical integration calculates the area under a given curve using a numerical approximation. The most basic methods are inspired by the definition of integration of a function $f(x)$ defined on an interval $[a, b]$:

$$\int_a^b f(x)dx = \lim_{h \to 0} \sum_{j=0}^{n} f(x_j)h$$

where $b - a = nh$.

If we consider the sum for a fixed $h$ rather than in the theoretical limit as $h$ goes to zero, then as $h$ becomes smaller the summation approximation becomes more accurate and this would be one method of numerical integration, otherwise called quadrature.

A general quadrature formula looks like this:

$$Q[f] = \sum_{j=0}^{n} w_j f(x_j) = w_0 f(x_0) + ... + w_n f(x_n)$$

where $a = x_0 < x_1 < x_2 < ... < x_n = b$.

If we approximate the integral with such a quadrature formula we have an associated error with the approximation (as we did with the numerical differentiation).

$$\int_a^b f(x)dx = Q[f] + E(f)$$

**Newton-Cotes Formulas** We begin by assuming that the function $f(x)$ can be approximated by a polynomial

$$P_n(x) = \sum_{j=0}^{n} a_j x^j$$

where the truncation error in this case would be proportional to the $(n+1)^{th}$ derivative

$$E(f) = Af^{(n+1)}(c)$$

where $A$ is constant. Given

$$x_j = x_0 + hj \qquad f_j = f(x_j)$$

We are able to then obtain several integration algorithms.

The Trapezoidal, Simpson's and Simpson's 3/8 rules are:

$$\begin{aligned}
\int_{x_0}^{x_1} f(x)dx &= \frac{h}{2}(f_0 + f_1) - \frac{h^3}{12}f''(c) \\
\int_{x_0}^{x_2} f(x)dx &= \frac{h}{3}(f_0 + 4f_1 + f_2) - \frac{h^5}{90}f^{(4)}(c) \\
\int_{x_0}^{x_3} f(x)dx &= \frac{3h}{8}(f_0 + 3f_1 + 3f_2 + f_3) - \frac{3h^5}{80}f^{(4)}(c)
\end{aligned}$$

There are several ways of deriving these methods, lets derive the trapezoid method here (see Kutz's notes for derivation of Simpson's rule).

A geometrical approach is as follows:

$$\begin{aligned}
\int_{x_0}^{x_1} &= \approx \text{area of trapezoid} \\
&= \frac{1}{2}(f(x_1) + f(x_0))(x_1 - x_0) \\
&= (x_1 - x_0)\frac{f(x_0) + f(x_1)}{2}
\end{aligned}$$

Derivation of the trapezoidal rule from Calculus beginning with the polynomial approximation to the function $f(x)$

$$f(x) \approx f_n(x) = \sum_{j=0}^{n} a_j x^j$$

where $f_n(x)$ is the $n^{th}$ order polynomial. The Trapezoidal method assumes $n = 1$:

$$\begin{aligned}
\int_{x_0}^{x_1} f(x)dx &= \int_{x_0}^{x_1} f_1(x)dx + E \\
&= \int_{x_0}^{x_1} \sum_{j=0}^{1} a_j x^j dx + E(f) \\
&= \int_{x_0}^{x_1} (a_0 + a_1 x)dx + E(f) \\
&= a_0(x_1 - x_0) + a_1 \left( \frac{x_1^2 - x_0^2}{2} \right) + E(f)
\end{aligned}$$

Now inserting the endpoint values

$$\begin{aligned} f(x_0) &= a_0 + a_1 x_0 \\ f(x_1) &= a_0 + a_1 x_1 \end{aligned}$$

Solving for $a_1$ and $a_2$:

$$\begin{aligned} a_1 &= \frac{f(x_1) - f(x_0)}{x_1 - x_0} \\ a_0 &= \frac{f(x_0)x_1 - f(x_1)x_0}{x_1 - x_0} \end{aligned}$$

Replacing $a_0$ and $a_1$ results in the Trapezoidal formula:

$$\int_{x_0}^{x_1} f(x)dx = (x_1 - x_0)\frac{f(x_0) + f(x_1)}{2} + E$$

You should note the order of accuracy as $O(h^2), O(h^4)$ and $O(h^4)$ for the Trapezoidal, Simpson's and Simpson's 3/8 methods.

$$\int_{x_0}^{x_1} f(x)dx = \frac{h}{2}(f_0 + f_1) - \frac{h^3}{12}f''(c)$$

**Composite Rules** are used in order to apply the integral formula over an interval $[a, b]$ which has been subdivided $(a = x_0 < x_1 < x_2 < ... < x_n = b)$, so we return to a more general quadrature formula

$$\begin{aligned}
\textstyle\int_a^b \; f(x)dx \approx Q_h[f] &= \sum_{j=0}^{n-1} \frac{h}{2}(f_j + f_{j+1}) \\
&= \frac{h}{2}(f_0 + f_1) + \frac{h}{2}(f_1 + f_2) + ... + \frac{h}{2}(f_{n-1} + f_n) \\
&= \frac{h}{2}(f_0 + 2f_1 + 2f_2 + ... + 2f_{n-1} + f_n) \\
&= \frac{h}{2}\left(f_0 + f_n + 2\sum_{j=1}^{n-1} f_j\right)
\end{aligned}$$

**Improving accuracy of the method** taking the composite rule for a step size of $2h$ instead of the original step size of size $h$:

$$\begin{aligned}
\int_a^b f(x)dx &\approx Q_{2h}[f] = \sum_{j=0}^{n/2-1} \frac{2h}{2}(f_{2j} + f_{2j+2}) \\
&= h(f_0 + f_2) + h(f_2 + f_4) + ... + h(f_{n-2} + f_n) \\
&= h(f_0 + 2f_2 + 2f_4 + ... + 2f_{n-2} + f_n) \\
&= h\left(f_0 + f_n + 2\sum_{j=1}^{n/2-1} f_{2j}\right)
\end{aligned}$$

Comparing this with $Q_h[F]$ we see that

$$Q_h[f] = \frac{1}{2}Q_{2h}[f] + h(f_1 + f_3 + ... + f_{n-1})$$

This shows that we can improve accuracy by halving the step size $h$ and this may be applied in order to obtain a desired accuracy.

Next week we'll cover the MATLAB built in functions. Here we implement the Trapezoidal method using the formula:

```
% Integrate the function x^2 over a domain
% x = [1,5] given a step size of h = .2
h = 0.2;

% Calculate the number of points needed
n = (5-1)/h;
x = (1+h:h:5);
f0 = 1^2;
f(1:n) = x(1:n).^2;
Trap = (h/2)*(f0+f(n)+2*sum(f(1:n-1)))

% Compare the result with the actual integral
(5^3-1^3)/3
```