Convergence of the Optimal Feedback Policies in a Numerical Method for a Class of Deterministic Optimal Control Problems [1]

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Abstract

We consider a Markov chain based numerical approximation method for a class of deterministic nonlinear optimal control problems. It is known that methods of this type yield convergent approximations to the value function on the entire domain. These results do not easily extend to the optimal control, which need not be uniquely defined on the entire domain. There are, however, regions of strong regularity on which the optimal control is well defined and smooth. Typically, the union of these regions is open and dense in the domain. Using probabilistic methods, we prove that on the regions of strong regularity, the Markov chain method yields a convergent sequence of approximations to the optimal feedback control. The result is illustrated with several examples.

1 Introduction

In this paper, we prove that an efficient Markov chain based numerical approximation method for a general class of nonlinear optimal control problems yields feedback controls which converge (on most of the domain) to the optimal feedback control for the problem that is being approximated. We consider an infinite time horizon problem on a finite domain in $\mathbb{R}^n$, with deterministic dynamics which are affine in the control variable. The running cost $L(x, u)$ is quadratic in the control variable u and is fully nonlinear in the state variable x, and there is no exit cost. Any problem in this class can be reduced by a simple change of variables to one with dynamics of Calculus of Variations type, and we find it convenient in our analysis to consider that form.

In general, one cannot explicitly evaluate either the value function or the optimal control, so accurate numerical approximation methods are needed. The quantity of interest for applications is typically the optimal control, and considerations of robustness in implementation make it important to have the optimal control in feedback form. Furthermore, for many recent applications, including robust control [3] and problems in computer vision [11, 20], approximations to the optimal feedback control and to the closely related gradient of the value function are needed. Given the fact that the control need not be uniquely defined, however, almost all of the literature focuses on approximating the value function, which, under our assumptions, is well defined and Lipschitz on the entire domain.

A natural class of numerical methods, first described by Kushner [18], involves replacing the limit control problem with an approximating problem whose state variable takes values on a finite grid. The deterministic dynamics are replaced by a Markov chain so that movement in an arbitrary direction can be approximated by appropriate probabilities of jumping to neighboring gridpoints. As the underlying grid is refined, the value function for the Markov chain control problem becomes an increasingly good approximation to the value function for the limit problem.

As we noted earlier, the optimal feedback control for the limit problem is not uniquely defined at all points, and this makes it difficult to construct an approximate optimal control on the entire domain. However, there are large subsets of the domain, called regions of strong regularity, on which it is uniquely defined, smooth, and in feedback form. Our main theorem states that the numerical method described in reference [5] yields a convergent sequence of approximations to the limit optimal feedback control in the regions of strong regularity.

We remark that our proof is applicable to a larger class of control problems than the one considered here. The quadratic structure of the running cost is not essential and can be replaced by suitable smoothness and convexity conditions. We restrict our attention to the quadratic case in order to streamline the presentation. Furthermore, the class of problems that we consider is important in that the infima in the discrete dynamic programming equation can be evaluated analytically, eliminating the need for computationally intensive numerical minimizations.

To our knowledge, there are no other general results of this type. Almost all of the literature, both probabilistic [5, 18, 19] and analytic using viscosity solution methods [1, 8], is dedicated to proving convergence of the value functions on the entire domain, and convergence of the controls does not follow naturally from those proofs. That is not surprising, since our proof strongly exploits smoothness properties which hold only in the regions of strong regularity. Some results regarding the convergence of controls in the Markov chain approximation method have been obtained in [12], but the situation there is quite specialized and is restricted to one dimension. In general, one dimensional problems are qualitatively easier to deal with because the control can only point in two directions, while the number of possible directions for $n \geq 2$ is uncountable. In fact, a simple calculation using the Dynamic Programming Equations in the present problem reveals the following startling result for the case where $n = 1$. For any point x at which the limit problem is regular and for a sufficiently refined grid, the optimal feedback control for the approximating problem is exactly equal to the limit value! This observation has limited practical value, but it does serve as a powerful reminder of the unique nature of one dimensional problems.

In our development, we draw liberally on ideas presented by Fleming in [14]. There, a similar problem is considered with a small variance Brownian Motion perturbation of the deterministic dynamics taking the place of the Markov chain approximations in our problem. In a future paper, we will apply the present result to obtain a full asymptotic expansion of the limit value function in the regions of strong regularity, analogous to the expansion obtained in [14]. Using this expansion, we will present a new numerical method which, under some additional assumptions, will be proved to yield approximations which are second order accurate in the regions of strong regularity.

The outline of this paper is as follows. In Section 2, we state our assumptions, introduce the limit optimal control problem, and define the regions of strong regularity on which our results will hold. Section 3 is dedicated to defining the approximating optimal control problems and their associated

Markov chain dynamics, while in Section 4 we establish some preliminary convergence results. The main theorem is stated and proved in Section 5, and we conclude in Section 6 with computational examples.

We end this section with some notation. For a vector $x \in I\!R^n$, $\|x\|$ is the Euclidean norm, $\|x\|_1 = \Sigma_{i=0}^n |x_i|$ is the l^1-vector norm, and $|x| = (|x_1|, \ldots |x_n|)$ is the componentwise absolute value. For a process $X(\cdot)$ taking values in $I\!R^n$ and for $S < +\infty$, $\|X(\cdot)\|_S = \int_0^S \|X(t)\| dt$ is the integrated L^2-norm, and $|\!|\!| X(\cdot) |\!|\!|_S = \sup_{0 \leq t \leq S} \|X(t)\|$ is the uniform L^2-norm. For any two subsets A and A' of $I\!R^n$, $d(A, A')$ denotes minimum Euclidean distance between $\overline{A}$ and $\overline{A'}$, while $B_\varepsilon(A)$ is the open ball of radius ε around A.

For a smooth function f mapping $I\!R^n$ to $I\!R$, $D_i f(x) = \frac{\partial}{\partial x_i} f(x)$, and the gradient is $Df(x) = (D_1 f(x), \ldots, D_n f(x))$. For $h > 0$, the operators $D^{h,\pm}$ are finite difference approximations to the gradient operator. So the ith component of $D^{h,+} f(x)$ is

$$D_i^{h,+} f(x) = \frac{f(x + he_i) - f(x)}{h},$$

while the ith component of $D^{h,-} f(x)$ is

$$D_i^{h,-} f(x) = \frac{f(x) - f(x - he_i)}{h}.$$

The positive part of a scalar is $a^+ = \max(a, 0)$, and its negative part is $a^- = -\min(a, 0)$. For a vector, the positive and negative parts are taken componentwise, so that $x^\pm = (x_1^\pm, \ldots, x_n^\pm)$.

2 Deterministic Control Problem

In this section we describe a deterministic optimal control problem on a bounded domain, with zero exit cost. Since our goal is to obtain the solution to this problem as the limit of numerical approximations, we will refer to it as the limit problem. Let $G \subset I\!R^n$ be open with compact closure, and assume that G satisfies uniform interior and exterior cone conditions (see [5] for definitions). Let b and c be C^∞ functions from $I\!R^n$ to $I\!R$, and let a be a C^∞ function from $I\!R^n$ to the space of symmetric positive definite $n \times n$ matrices. Notice that a is uniformly positive definite on G. Assume that $c(x) \geq c_0 > 0$ on G. For a control $\underline{u}^0(t)$ which is in $L^2([0, S]; I\!R^n)$ for all $S < +\infty$ and for an initial condition $x \in G$, we define $\underline{X}^0(t)$ by the dynamics

$$\underline{X}^0(t) = x + \int_0^t \underline{u}^0(s) ds, \qquad (2.1)$$

up to the time when it exits from the domain G. We define the exit time $\underline{\tau}^0 = \inf[t : \underline{X}^0(t) \notin G]$. For the running cost

$$L(x,u) = \frac{1}{2}\Big\langle (u - b(x)), a^{-1}(x)(u - b(x))\Big\rangle + c(x),$$

we define the payoff functional

$$J^0(x, \underline{u}^0) = \int_0^{\underline{\tau}^0} L(\underline{X}^0(t), \underline{u}^0(t))dt.$$

The problem is to minimize the payoff by choosing a suitable control. Define the value function,

$$V^0(x) = \inf_{\underline{u}^0} J^0(x, \underline{u}^0),$$

where the infimum is over controls $\underline{u}^0$ which are in $L^2([0,S]; I\!\!R^n)$ for all $S < +\infty$. We employ the underscore notation here to indicate trajectories which are obtained from an arbitrary control. The same notations, without the underscores, will be used later to refer to trajectories which are obtained through the application of an optimal control.

We note that our analysis subsumes a much larger class of deterministic control problems. Namely, any problem with smooth dynamics which depend affinely on the control variable u and with a cost structure of the type described above can be made to fit within our framework by a simple change of variables.

The dynamics in (2.1) involve an open loop control $\underline{u}^0(t)$ which is defined for all $t > 0$. It is generally desirable, from the point of view of robustness and for convenience of implementation, to consider controls which can be represented in the feedback form,

$$\underline{X}^0(t) = x + \int_0^t \underline{u}^0(\underline{X}^0(s))ds. \tag{2.2}$$

A key feature of the regions of strong regularity is that the optimal open loop controls for all initial conditions in a region of strong regularity correspond to a unique smooth feedback function $u^0(x)$. That is the quantity that we wish to approximate.

The following lemma allows us to regard the limit control problem as one with a finite time horizon and a compact control space, when it is convenient to do so. Thus, it follows from [2, Theorem 6.1] that V^0 is the unique nonnegative viscosity solution on G to the Dynamic Programming Equation (DPE)

$$\inf_u \Big[\langle u, DV^0(x)\rangle + L(x,u)\Big] = 0, \tag{2.3}$$

with the continuous boundary condition $V^0(x) = 0$ on ∂G. See references [1] and [15] for a thorough account of the relationship between viscosity solutions of Hamilton-Jacobi PDE's and the value functions for various types of optimal control problems.

Lemma 2.1 *$V^0(x)$ is bounded and uniformly Lipschitz on G, and there exists $T < +\infty$ such that every optimal trajectory exits from G by time $T-1$. Furthermore, there exists $U^0 < +\infty$ such that the norm of every optimal open loop control is bounded by U^0 for each $0 \leq t \leq T-1$.*

Proof. The Lipschitz property for $V^0(x)$ is a consequence of the controllability implied by the fact that $a(x)$ is uniformly positive definite. Since it is possible to move with unit velocity in any direction with bounded running cost, the principle of optimality implies for all $x, y \in G$ the relation $V^0(x) \leq V^0(y) + C\|y - x\|$ for some fixed $C < +\infty$, and this implies a uniform Lipschitz property. The bound on $V^0(x)$ follows from the Lipschitz property, and the bound on the exit times is then implied by the lower bound $c_0 > 0$ on the running cost.

To obtain the bound on the optimal controls, it suffices to find $U^0 < +\infty$ such that any control $\underline{u}^0(t)$ which has norm exceeding U^0 on some measurable set $A \subset [0, T-1]$ can be replaced by one with a smaller maximum norm, resulting in a lower cost. Since the running cost $L(x, u)$ is uniformly convex in the control variable u, we can accomplish this for U^0 sufficiently large by using $\underline{u}^0(t)/2$ on a version of the set A stretched by a factor of two. This results in following the same trajectory at a slower speed, and a straightforward calculation indicates that it yields a lower cost. ■

It turns out that V^0 is smooth on most of the domain G. Let Q be a relatively open subset of $\overline{G}$. We call Q a region of strong regularity if the following hold:

1. For each initial condition $x \in Q$, there is a unique optimal open loop control, and the corresponding trajectory is contained in Q up to its exit time. The optimal trajectory meets ∂G nontangentially.
2. $V^0 \in C^\infty(Q)$.
3. There is a unique $u^0 \in C^\infty(Q)$ such that the optimal control can be represented in feedback form and is given by $u^0(x)$ for each $x \in Q$.

For a discussion of the classical method of characteristics and its application to proving the existence of regions of strong regularity for the present problem, see the appendices in [14] and [16]. Detailed information on the structure of the singularity sets for closely related problems can be found in references [13], [7], and [6]. In general, the union of the regions of strong regularity is open and dense in the domain. Since V^0 is a classical solution to the DPE (2.3) on the regions of strong regularity, the optimal feedback control can be explicitly evaluated there,

$$u^0(x) = -a(x)DV^0(x) + b(x). \tag{2.4}$$

Let B_0 be a subset of $\overline{G}$ such that $\overline{B_0} \subset Q$, and consider a nested sequence of four regions of strong regularity B, M, N, and Q such that

$$\overline{B_0} \subset B \subset \overline{B} \subset M \subset \overline{M} \subset N \subset \overline{N} \subset Q.$$

The main convergence results will be stated in terms of uniform limits on the set B_0. We assume the following:

Assumption 2.2 *The boundary section $Q \cap \partial G$ is parallel to one of the coordinate hyperplanes. Furthermore, the minimum distance in the outward normal direction from $Q \cap \partial G$ to $\partial G/Q$ is equal to $\tilde{\delta} > 0$.*

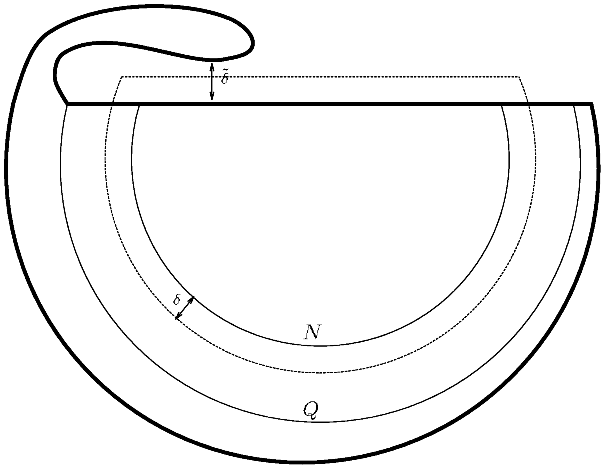

Figure 1: Region for Smooth Extension of u^0

It is convenient to have $u^0(x)$ defined and Lipschitz on all of $I\!R^n$, so we abuse notation by extending $u^0(x)$ to $I\!R^n$ and changing its values on the complement of $\overline{N}$. Let $\delta > 0$ be such that $\delta < d(N, \partial Q \cap G)$ and such that $\delta \leq \tilde{\delta}$, where $\tilde{\delta}$ is as in Assumption 2.2; see Figure 1. We can define a Lipschitz function $\tilde{u}^0(x)$ on $B_\delta(\overline{N})$ by setting $\tilde{u}^0(x) = u^0(x)$ on $B_\delta(\overline{N}) \cap G$ and by extending it to be constant across the boundary section $Q \cap \partial G$. Now let $\phi(x)$ be a C^∞ function on $I\!\!R^n$ taking values in $[0,1]$ such that $\phi(x) = 1$ on $B_{\delta/2}(\overline{N})$ and $\phi(x) = 0$ outside of $B_\delta(\overline{N})$. Such a function can be constructed by standard methods using a smooth convolution kernel [17, Theorem 0.17]. We can now redefine $u^0(x)$ to be equal to $\phi(x)\tilde{u}^0(x)$ on $B_\delta(\overline{N})$ and zero everywhere else. This new $u^0(x)$ is Lipschitz on $I\!\!R^n$ and satisfies equation (2.4) on the region N. Furthermore, $\|u^0(x)\| \leq U^0$ for each $x \in I\!\!R^n$ where $U^0 < +\infty$ is the bound from Lemma 2.1.

For any $x \in N$, let $X_x^0(t)$ be the trajectory obtained by applying the optimal feedback control u^0 with initial condition x. Since we use the extended version of u^0, we can define $X_x^0(t)$ by (2.2) for all $t \geq 0$. Let τ_x^0 be the first exit time of $X_x^0(t)$ from G, and let z_x^0 be its exit location. Notice that the definition of regions of strong regularity implies that $z_x^0 \in N$ for each $x \in N$, and that τ_x^0 is also the first exit time from the interior of N. We will often suppress the initial conditions in the subscripts of these notations.

Lemma 2.3 *For each sufficiently small $\varepsilon > 0$, there exists $\eta > 0$ such that the following holds. Let X be a trajectory with initial condition in N, and let τ_N and z_N be its exit time and location from the interior of N. If $|\!|\!| X - X_x^0 |\!|\!|_T \leq \eta$ holds for some $x \in M$, then $|\tau_N - \tau_x^0| \leq \varepsilon$ and $\|z_N - z_x^0\| \leq \varepsilon$.*

Proof. Recall from Lemma 2.1 the bounds T and U^0 on the exit times and on the optimal controls, respectively. Given the way we extended u^0 beyond N and given the nontangential exit property for the regions of strong regularity, we have that

$$X_x^0(t) \in B_{\delta/2}(N)/G$$

for all $\tau_x^0 \leq t \leq \tau_x^0 + \Delta$, where $\Delta = \min(1, \delta/2U^0)$. Furthermore, there is $0 < \gamma \leq \delta/2$ such that the component of $\dot{X}_x^0(t)$ in the outward normal direction away from the boundary segment $Q \cap \partial G$ is at least equal to γ for $\tau_x^0 \leq t < \tau_x^0 + \Delta$. On account of the second part of Assumption 2.2, it follows that

$$d(X_x^0(t), G) \geq \gamma(t - \tau_x^0)$$

for $\tau_x^0 \leq t < \tau_x^0 + \Delta$; see Figure 1. Thus, if $|\!|\!| X - X_x^0 |\!|\!|_T < \gamma\varepsilon$, then $\tau_N \leq \tau_x^0 + \varepsilon$. For the remainder of this proof, we consider only those η such that $\eta \leq \gamma\varepsilon$, so we may assume that $\tau_N \leq \tau_x^0 + \varepsilon$.

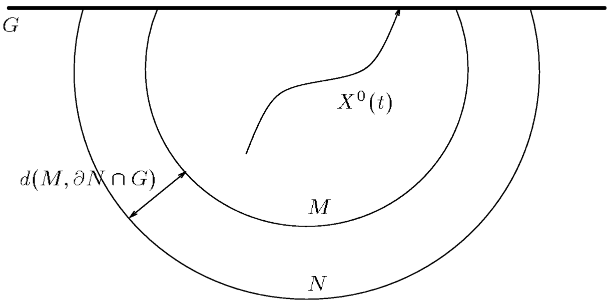

Figure 2: Regions of Strong Regularity

To establish the lower bound for τ_N, we begin by observing that the nontangential exit property for regions of strong regularity implies

$$d(X_x^0(t), \partial N \cap G) \geq d(M, \partial N \cap G) - \varepsilon U^0$$

for all $0 \leq t \leq \tau_x^0 + \varepsilon$; see Figure 2. Thus, if $\varepsilon > 0$ is sufficiently small and if

$$\| X - X_x^0 \|_T < d(M, \partial N \cap G) - \varepsilon U^0,$$

then it follows that $X(\tau_N) \in \partial G$. We observe that $\tau_y^0 \leq C d(y, \partial G)$ holds for any $y \in G$, where $C = \sup_{x \in G} V^0(x)/c_0$ and $c_0 > 0$ is the lower bound on the running cost. Thus, if $\| X - X_x^0 \|_T < \varepsilon / C$, then the previous display implies

$$\tau_x^0 \leq \tau_N + C d(X_x^0(\tau_N), \partial G) \leq \tau_N + \varepsilon.$$

We have shown that $|\tau_N - \tau_x^0| \leq \varepsilon$ is satisfied for sufficiently small $\eta > 0$. Now, we observe that

$$\begin{aligned} \|z_N - z_x^0\| &\leq \|X(\tau_N) - X_x^0(\tau_N)\| + \|X_x^0(\tau_N) - X_x^0(\tau_x^0)\| \\ &\leq \eta + U^0 |\tau_N - \tau_x^0|. \end{aligned}$$

Thus, we can use the above argument to select a possibly smaller $\eta > 0$ such that $|\tau_N - \tau_x^0| < (\varepsilon - \eta)/U^0$, and so establish the bound $\|z_N - z_x^0\| \leq \varepsilon$. ■

The following lemma deals with the continuity of the trajectories with respect to the initial condition. It will be useful in establishing uniformity in the pathwise convergence results of Section 4.

Lemma 2.4 *Let* $x_k \in M$ *be such that* $x_k \to x \in M$*, and let* T *be as in Lemma 2.1. Then*

$$\||| X^0_{x_k} - X^0_x |||_T, \quad \|u^0(X^0_{x_k}) - u^0(X^0_x)\|_T, \quad |\tau^0_{x_k} - \tau^0_x|, \quad \textit{and} \quad \|z^0_{x_k} - z^0_x\|$$

all converge to zero as $k \to \infty$*.*

Proof. Since the vector field u^0 is globally Lipschitz, the convergence of $||| X^0_{x_k} - X^0_x |||_T$ to zero can be established by a routine application of Gronwall's inequality to the dynamics in (2.2). Given that, the remaining parts of the lemma follow from the uniform continuity of the feedback control u^0 and from Lemma 2.3. ■

3 Markov Chain Approximations

We employ the method of approximating Markov chains developed by Kushner [18] to compute approximate solutions to the deterministic optimal control problem described above. For an up to date treatment of this subject, see the book of Kushner and Dupuis [19]. Our approximation is essentially the one used by Boué and Dupuis in [5]. In order to numerically approximate the value function V^0 and the optimal control u^0, we need to define a process which takes values on a finite lattice and which approximates the continuous dynamics. We circumvent the problem of only being able to move in the lattice directions by introducing jump probabilities which give rise to arbitrary mean velocities. The value function corresponding to this process, with the same cost structure as above, satisfies on the lattice a DPE analogous to (2.3). Thus, it is possible to numerically compute the value function and the optimal feedback control for the approximating problem. We will show that, at least in the compact set B_0, these are good approximations to V^0 and u^0.

Let $h > 0$ be a discretization parameter, and define the discrete domain $G^h = h\mathbb{Z}^n \cap G$. For any $A \subset \mathbb{R}^n$, we define $A^h = h\mathbb{Z}^n \cap A^o$, where A^o is the interior of A. We consider limits as $h \to 0$, with the h chosen such that the hyperplane in which the boundary section $Q \cap \partial G$ lies lines up with the lattice $\mathbb{Z}^h$ (see Assumption 2.2). We will construct a continuous time controlled jump Markov process on G^h which approximates the deterministic dynamics in (2.2). This process will give rise to the same DPE obtained in [5] by using a discrete time Markov chain. For our purposes, however, it is more convenient to work with a continuous time jump Markov process.

Let $\underline{u}^h$ be any feedback control on G^h and extend $\underline{u}^h$ to be equal to u^0 on $\mathbb{Z}^h/G^h$. Let $\underline{X}^h$ be the Markov process with controlled generator given by

$$\mathcal{L}_u^h f = \langle u^+, D^{h,+} f\rangle - \langle u^-, D^{h,-} f\rangle \tag{3.1}$$

for any smooth function f mapping $\mathbb{R}^n$ to $\mathbb{R}$. See Section 1 for the notation in this definition. The stochastic dynamics corresponding to this generator will be called the h-dynamics. As in the description of the limit problem, we employ the underscore notation to indicate objects which are obtained from the application of an arbitrary control.

Since we consider only feedback controls, it is straightforward to construct $\underline{X}^h$, as in Section 4.3 of [19] and in [9]. We define a sequence of i.i.d. exponential random fields parameterized by u, with mean values specified as follows:

$$\overline{\Delta t}^h(u) = \begin{cases} \frac{h}{\|u\|_1} & u \neq 0 \\ h & u = 0. \end{cases}$$

Suppose that after $m-1$ jumps, $\underline{X}^h(s)$ is defined for $0 \le s \le t$ and that $\underline{X}^h(t) = x$. Then we take $\underline{X}^h(s) = x$ for all $t \le s < t+\eta$, where the waiting time η is the exponential random variable obtained by evaluating the mth random field with the parameter value $u = \underline{u}^h(\underline{X}^h(s))$. If $u = 0$ then $\underline{X}^h(t+\eta) = x$, but otherwise it is conditionally distributed according to the jump probabilities,

$$p^h(x, y|u) = \begin{cases} \frac{u_i^\pm}{\|u\|_1} & \text{if } y = x \pm he_i \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to verify that the mean velocity of $\underline{X}^h$ at time t conditioned on $\underline{X}^h(t) = x$ is equal to $\underline{u}^h(x)$, so this is a consistent approximation to the limit dynamics in (2.2).

We consider the semi-martingale decomposition of $\underline{X}^h$. For a given feedback control $\underline{u}^h$ and fixed initial condition $x \in G^h$, we write

$$\underline{X}^h(t) = \underline{Y}^h(t) + \underline{m}^h(t), \tag{3.2}$$

where the stochastic process $\underline{Y}^h$ is defined w.p.1 by

$$\underline{Y}^h(t) = x + \int_0^t \underline{u}^h(\underline{X}^h(s))ds.$$

The consistency of the jump dynamics guarantees that $\underline{m}^h(t)$ is a local martingale with mean zero. Furthermore, the variance of $\underline{m}^h(t)$ is controlled by the parameter h. That is the content of the following lemma.

Lemma 3.1 *Fix $h > 0$, and let $\underline{u}^h$ be any feedback control which satisfies $\|\underline{u}^h(x)\| \leq K < +\infty$ for all $x \in h\mathbb{Z}^n$. Then the bound*

$$E_x\|\underline{m}^h(\sigma)\|^2 \leq hKE_x\sigma,$$

holds for any bounded stopping time σ.

Proof. The triple $(\underline{m}^h, \underline{X}^h, \underline{Y}^h)$ is Markov and measurable with respect to the σ-algebra generated by $\underline{X}^h$. We consider its generator $\tilde{\mathcal{L}}^h$. Since $\underline{m}^h = \underline{X}^h - \underline{Y}^h$, we have

$$\tilde{\mathcal{L}}^h f = \langle \underline{u}^{h,+}(\underline{X}^h), D^{h,+}f\rangle - \langle \underline{u}^{h,-}(\underline{X}^h), D^{h,-}f\rangle - \langle \underline{u}^h(\underline{X}^h), Df\rangle,$$

for any smooth function of the form $f(m,x,y) = f(m)$. Given the fact that $\underline{m}^h(t)$ takes values in a bounded set for bounded values of $t \geq 0$, the general theory of piecewise deterministic process [9, Theorem 5.5] implies that for any smooth function f on $\mathbb{R}^n$, for any initial condition $x \in G$, and for any bounded stopping time σ,

$$E_x\left[f(\underline{m}^h(\sigma)) - f(0) - \int_0^\sigma \tilde{\mathcal{L}}^h f(\underline{m}^h(t))dt\right] = 0. \tag{3.3}$$

Taking $f(m) = \|m\|^2$, we use (3.3) and the fact that for this choice of f,

$$|\tilde{\mathcal{L}}^h f(\underline{m}^h(s))| \leq 2h\|\underline{u}^h(\underline{X}^h(s))\|_1$$

to obtain

$$E_x\|\underline{m}^h(\sigma)\|^2 \leq 2hE_x\int_0^\sigma \|\underline{u}^h(\underline{X}^h(s))\|_1 ds.$$

Given the bound on $\|\underline{u}^h\|$, this completes the proof. ■

We now formulate the discrete approximation to the optimal control problem discussed in Section 2. Define the value function

$$V^h(x) = \inf_{\underline{u}^h} E_x \int_0^{\underline{\tau}^h} L(\underline{X}^h(t), \underline{u}^h(\underline{X}^h(t)))dt, \tag{3.4}$$

where the exit time is $\underline{\tau}^h = \inf[t : \underline{X}^h(t) \notin G^h]$, and the infimum is over feedback controls $\underline{u}^h$. Using standard methods [19, Section 4.3] it can be shown that V^h is the unique solution on G^h to the DPE

$$\inf_u\left[\left\langle u^+, D^{h,+}V^h(x)\right\rangle - \left\langle u^-, D^{h,-}V^h(x)\right\rangle + L(x,u)\right] = 0, \tag{3.5}$$

with zero boundary condition on $h\mathbb{Z}^n/G^h$. It is straightforward to verify that the (3.5) is equivalent to

$$V^h(x) = \inf_u \left[\sum_{y \in \mathbb{R}^n} p^h(x, y|u) V^h(y) + \overline{\Delta t}^h(u) L(x, u) \right], \tag{3.6}$$

and that the minimizing values of u are the same for these two equations. As suggested by the form of (3.6), the fixed point and an optimal feedback control can be found numerically using either Jacobi or Gauss-Seidel iteration schemes. We note that equation (3.6) is the DPE for a different approximating control problem, where a discrete time Markov chain is used to approximate the deterministic dynamics. That is the approach taken in [5], where the time step $\overline{\Delta t}^h(u)$ is used to interpolate the Markov chain into continuous time. As discussed in [5], the choice of one-sided transition probabilities and of a control dependent time step facilitates rapid convergence of the iterative schemes used to solve (3.6), and the required infima at each step can be evaluated analytically.

The DPE (3.5) gives rise to an optimal feedback control u^h on G^h. It is convenient, at this point, to abuse notation and to redefine u^h to be equal to u^0 on $h\mathbb{Z}^n/N^h$. Then, for each initial condition $x \in N^h$, there is a unique process X^h_x defined for all $t \geq 0$ which is optimally controlled by u^h until it exits from N^h. We define the exit time $\tau^h_{x,N} = \inf[t : X^h_x(t) \notin N^h]$ and the exit location $z^h_{x,N} = X^h_x(\tau^h_{x,N})$. Recall that N^h is defined to be $h\mathbb{Z}^n \cap N^o$, where N^o is the interior of N. Let m^h_x be the martingale part of the decomposition for X^h_x given by (3.2). As in the limit problem, we will often suppress the initial conditions in the subscripts of all of these notations.

The following remark and lemma simplify some of the analysis by allowing us to consider a compact domain and a compact control space.

Remark 3.2 We extended $u^0(x)$ to all of $\mathbb{R}^n$ in such a way that it is equal to zero off of the neighborhood $B_\delta(N)$. Thus, the same is now true of $u^h(x)$. Consequently, for all initial conditions $x \in N$, the trajectories X^0_x and X^h_x never leave the closed neighborhood $\overline{B_{\delta+h}(N)}$.

Lemma 3.3 *There exists a compact set* $U \subset \mathbb{R}^n$ *such that the extended optimal feedback controls* $u^0(x)$ *and* $u^h(x)$ *take values in* U *for all* $h > 0$ *and for all* $x \in \mathbb{R}^n$ *on which they are defined. Furthermore, the value functions* $V^0(x)$ *and* $V^h(x)$ *are bounded, uniformly in* h *and* x.

Proof. Recall from Lemma 2.1 that we obtained $U^0 < +\infty$ such that $\|u^0(x)\| \leq U^0$ for all $x \in I\!R^n$. From the DPE (3.5), it follows that for $x \in G^h$, each component of $u^h(x)$ is either equal to zero or is given by a bounded linear functional of $D^{h,\pm}(x)$. Thus, in order to find the set U, it suffices to establish a bound on $D^{h,\pm}V^h(x)$ which is uniform for all $h > 0$ and $x \in G^h$.

Without loss of generality, we consider the case of bounding $D_i^{h,+}V^h(x)$. The principle of optimality implies that the minimal cost starting at $x \in G^h$ can be no larger than the minimal cost starting at $x + he_i$ plus the expected cost of getting from x to $x+he_i$ under any suboptimal control. The fact that a is uniformly positive definite while b and c are bounded implies that there exists a constant $K < \infty$ such that $L(x,u) \leq K$ whenever $\|u\| = 1$. Taking $u = he_i$, we obtain $V^h(x) \leq V^h(x+he_i)+KE_x\eta$, where the waiting time η is exponential with mean h. Thus, we have shown $V^h(x) \leq V^h(x+he_i)+hK$. The reverse inequality is established by using $x+he_i$ as the initial condition, and it follows that $|D_i^{h,+}V^h(x)| < K$. That concludes the proof for the optimal controls. A uniform bound on the value functions $V^h(x)$ follows from the above argument and from the boundedness of the domain G. Along with the bound on $V^0(x)$ from Lemma 2.1, this finishes the proof. ■

4 Preliminary Convergence Results

The main results of this section are contained in Lemma 4.1. It states that in the region of strong regularity B, the optimal trajectories, open loop controls, exit times, and exit locations converge in probability, uniformly with respect to initial conditions. Since the limit objects are deterministic, we are able to use convergence in distribution arguments to establish the desired convergence in probability. Uniformity with respect to initial conditions is a consequence of the continuity properties in Lemma 2.4.

Lemma 4.1 *For every $\varepsilon > 0$, there exists $h_0 > 0$ such that for all $0 < h \leq h_0$ and for all initial conditions $x \in B^h$,*

$$(i) \qquad P_x\left[\||\, X_x^h - X_x^0 \,\||_T > \varepsilon\right] < \varepsilon$$

$$(ii) \qquad P_x\left[\|u^h(X_x^h) - u^0(X_x^0)\|_T > \varepsilon\right] < \varepsilon$$

$$(iii) \qquad P_x\left[|\tau_{x,N}^h - \tau_x^0| > \varepsilon\right] < \varepsilon$$

$$(iv) \quad P_x\left[\|z^h_{x,N} - z^0_x\| > \varepsilon\right] < \varepsilon$$

where T is the bound on the exit times from Lemma 2.1.

We will use the following convergence result [5]. In fact, we will repeat part of the argument to prove this theorem in our proof of Lemma 4.1, but the exposition is made more transparent by assuming convergence of the value functions.

Theorem 4.2 **(i)** *Let $x^h \in G^h$ for $h > 0$ be such that $x^h \to x \in G$ as $h \to 0$. Then $V^h(x^h) \to V^0(x)$ as $h \to 0$.* **(ii)** *For any $\varepsilon > 0$, there exists $h_0 > 0$ such that $|V^h(x) - V^0(x)| < \varepsilon$ for all $0 < h \le h_0$ and all $x \in G^h$.*

Proof. Part (i) is proved, in a somewhat more general setting, as Theorem 5.4 in [5]. If part (ii) is false, then there is $\varepsilon > 0$ and a sequence $x^h \in G^h$ with $h \to 0$ such that $|V^h(x^h) - V^0(x^h)| > \varepsilon$ for each h. Since $\overline{G}$ is compact and V^0 is uniformly continuous on $\overline{G}$, we can extract a subsequence such that $x^h \to x \in \overline{G}$ and $|V^h(x^h) - V^0(x)| > \varepsilon/2$ for each h, which contradicts part (i) of the theorem. ■

To facilitate treating the optimal trajectories and controls in the framework of convergence in distribution, we adopt some standard definitions. We treat the processes X^h as random variables taking values in $\mathcal{D}([0,\infty); \mathbb{R}^n)$, the space of $\mathbb{R}^n$-valued functions that are continuous from the right and have limits on the left. With the Skorokhod metric, $\mathcal{D}([0,\infty); \mathbb{R}^n)$ is a complete separable metric space [4], and convergence of a sequence in $\mathcal{D}([0,\infty); \mathbb{R}^n)$ is equivalent to convergence of that sequence in $\mathcal{D}([0,S]; \mathbb{R}^n)$ for each $S < +\infty$. If a sequence in $\mathcal{D}([0,\infty); \mathbb{R}^n)$ converges to a continuous function under the Skorokhod metric, then it also converges in the uniform norm $\|\!| \cdot \|\!|_S$ for each $S < +\infty$.

We also consider the space of relaxed controls. A relaxed control is an element of $\mathcal{R}(U \times [0,\infty))$, the space of all Borel measures ν on $U \times [0,\infty)$ such that $\nu(\mathbb{R}^n \times [0,S]) = S$ for each $S \le +\infty$, where $U \subset \mathbb{R}^n$ is the compact control set from Lemma 3.3. This space can be metrized as a complete separable metric space, with a metric such that $\nu_k \to \nu$ if and only if the restriction of ν_k to $U \times [0,S]$ converges weakly to the restriction of ν to $U \times [0,S]$ for all $S \le +\infty$ [19, Section 9.5]. The second marginal of any measure $\nu \in \mathcal{R}(U \times [0,\infty))$ is Lebesgue measure, so the decomposition $\nu(du \times dt) = \nu_t(du)dt$ holds, where ν_t is a probability measure for each $t \ge 0$. If ν is a random variable, then this decomposition can be done so that it holds almost surely and so that for all $t \ge 0$, ν_t is a random variable. We note that the following version of Fatou's Lemma holds [4, Theorems 5.1 and 5.3]. If $\nu_k \to \nu$, then

$$\liminf_{k\to\infty} \int_{U\times[0,S]} f d\nu_k \geq \int_{U\times[0,S]} f d\nu, \tag{4.1}$$

for any $S < +\infty$ and for any continuous non-negative function f on the space $U \times [0,\infty)$.

For $h > 0$ and for each $t \geq 0$ let $\nu_t^h = \delta_{u^h(X^h(t))}$, where δ_u is the probability measure on U that places unit mass at the point u, and u^h is the optimal feedback control for the prelimit problem with parameter h. The corresponding optimal relaxed control is the measure valued random variable given by $\nu^h(A \times A') = \int_{A'} \nu_t^h(A)dt$ for Borel sets $A \subset U$ and $A' \subset [0,\infty)$. In terms of the optimal relaxed control measures, the inequality

$$V^h(x) \geq E_x \int_{U\times[0,\tau_N^h]} L(X^h(t), u)\nu^h(du \times dt) \tag{4.2}$$

holds for each $x \in N^h$. Equality may not hold in (4.2) because it is possible to have $X^h(\tau_N^h) \notin \partial G$; for equality to hold, we would need to add $E_x V^h(X^h(\tau_N^h)) \geq 0$ to the right hand side of expression (4.2). For initial conditions in the region of strong regularity N, we can similarly define ν^0 to be the measure in $\mathcal{R}(U \times [0,\infty))$ with first marginals $\nu_t^0 = \delta_{u^0(X^0(t))}$. Since ν^0 is not a random variable, an analogue to equation (4.2) holds for V^0 without the expectation:

$$V^0(x) = \int_{U\times[0,\tau^0]} L(X^0(t), u)\nu^0(du \times dt). \tag{4.3}$$

Notice that the inequality in (4.2) is replaced by equality in (4.3) because τ^0 is the exit time of X^0 from G. The proof of the following lemma is nearly identical to the proof of Lemma 5.3 in [5]. The only necessary modification is to use the martingale estimate from Lemma 3.1 in place of an analogous estimate obtained by applying a standard conditioning argument to the discrete time processes in [5].

Lemma 4.3 *For $h > 0$, let $x^h \in B^h$ be such that $x^h \to x \in \overline{B}$ as $h \to 0$. Then, using these initial conditions, the random variables (X^h, ν^h) are tight. Furthermore, for any subsequence along which the limit,*

$$(X^h, \nu^h) \to (X, \nu),$$

holds in the sense of distributions,

$$X(\cdot) = x + \int_0^{\cdot} \int_U u\, \nu_s(du)ds \tag{4.4}$$

is valid w.p.1.

Given the tightness from Lemma 4.3 and the convergence of the value functions from Theorem 4.2, we can use the uniqueness of optimal trajectories in regions of strong regularity for the limit problem to prove that the optimal trajectories and controls converge in distribution to the the appropriate limit quantities. That is the conclusion of the next lemma.

Lemma 4.4 *Let $x^h \in B^h$ be such that $x^h \to x \in \overline{B}$ as $h \to 0$. Then, using these initial conditions, the limit $(X^h, \nu^h) \to (X^0, \nu^0)$ holds in the sense of distributions as $h \to 0$.*

Proof. We consider the τ_N^h as random variables taking values in the compactified space $[0, \infty]$. Then, Lemma 4.3 implies that the random variables (X^h, ν^h, τ_N^h) are tight. Thus, given the continuity of the process in expression (4.4), for any subsequence there is a further subsequence along which the weak convergence $(X^h, \nu^h, \tau_N^h) \to (X, \nu, \tilde{\tau})$ holds for some limit random variable taking values in

$$\mathcal{C}([0, \infty); I\!R^n) \times \mathcal{R}(U \times [0, \infty)) \times [0, \infty].$$

We will show that for any such limit, (X, ν) is w.p.1 equal to (X^0, ν^0).

By the Skorokhod Representation Theorem [10], we can consider a probability space on which the convergence is w.p.1. Since the limit trajectory X is continuous, the convergence $X^h \to X$ is uniform on compact intervals. Thus, it is easy to verify that w.p.1 $\tilde{\tau} \geq \tau_N$, where τ_N is the first exit time of $X(t)$ from the interior of N. We obtain the following series of inequalities, each line of which is explained after the display:

$$\begin{aligned}
V^0(x) &= \lim_{h \to 0} V^h(x^h) \\
&\geq \lim_{h \to 0} E_{x^h} \int_{U \times [0, \tau_N^h)} L(X^h(t), u) \nu^h(du \times dt) \\
&\geq E_x \int_{U \times [0, \tilde{\tau})} L(X(t), u) \nu(du \times dt) \\
&\geq E_x \int_{U \times [0, \tau_N)} L(X(t), u) \nu(du \times dt) \\
&\geq E_x \int_0^{\tau_N} L(X(t), \dot{X}(t)) dt \\
&\geq V^0(x).
\end{aligned}$$

The first line is due to part (i) of Theorem 4.2; the second line comes from the representation in (4.2); the third line is obtained by applying (4.1) along with the standard version of Fatou's Lemma; the fourth line uses the fact that w.p.1 $\tilde{\tau} \geq \tau_N$; the fifth line follows from Jensen's inequality and the relation (4.4); and the final line is a consequence of the definition of $V^0(x)$.

Evidently, all of the inequalities in the previous display are in fact equalities. Thus, given the uniqueness of optimal trajectories in the regions of strong regularity, the last line implies that w.p.1, $X(t) = X^0(t)$ for $0 \leq t \leq \tau_N = \tau^0$. Recall that for equality to occur in Jensen's inequality with a strictly convex function, the probability measure must be a point mass. Thus, equality in the fifth line implies that w.p.1 $\nu_t = \nu_t^0$ for a.e. $0 \leq t \leq \tau^0$. It remains to show that w.p.1, $X(t) = X^0(t)$ and $\nu_t = \nu_t^0$ for a.e. $\tau^0 \leq t \leq T$.

Given that X^h is close to X^0 up to time τ^0, we can use the optimality of X^h, along with a uniform Lipschitz type bound on the V^h (see Lemma 3.3) and the lower bound on the running cost, to conclude that $\tau_N^h - \tau^0$ is small with arbitrarily high probability. Since the optimal controls are bounded, it follows that $\|X^h(t) - X^0(t)\|$ is arbitrarily small up to time $\tau^0 \vee \tau_N^h$, with high probability. This extends the w.p.1 equality of $X(t)$ and $X^0(t)$ up to time $\tau^0 \vee \tau_N^h$. Furthermore, since $u^h = u^0$ outside of N, Gronwall's inequality implies that $\|X^h(t) - X^0(t)\|$ converges to zero in probability for each $\tau^0 \vee \tau_N^h \leq t \leq T$, so that $X(t) = X^0(t)$ w.p.1 for all such t. This depends on the fact that the extended optimal controls u^0 and u^h point away from the region N near the boundary section $N \cap \partial G$ and that the h-dynamics are one sided, so that whenever X^h exits from N at a point in $N \cap \partial G$ (which happens with high probability), it reaches the region where $u^h = 0$ before returning to N; see Figure 1. Finally, $\nu_t = \nu_t^0$ for a.e. $\tau^0 \leq t \leq T$ follows from the above argument since $u^h = u^0$ outside of N and u^0 is uniformly Lipschitz on $\mathbb{R}^n$. ■

Proof of Lemma 4.1. Suppose that part (i) of the lemma is false. Then there exists $\varepsilon > 0$ along with a sequence $x^h \in B^h$ with $h \to 0$ such that $P\left[\||\, X_{x^h}^h - X_{x^h}^0 \,\||_T > \varepsilon\right] \geq \varepsilon$ for each h. Using the continuity of X^0 as a function of its initial condition from Lemma 2.4, we can extract a subsequence such that $x^h \to x \in \overline{B}$ and $P\left[\||\, X_{x^h}^h - X_x^0 \,\||_T > \varepsilon/2\right] \geq \varepsilon$ for each h. This is a contradiction, since the convergence in distribution of $X_{x^h}^h$ to the deterministic limit X_x^0 in Lemma 4.4 implies that $\||\, X_{x^h}^h - X_x^0 \,\||_T \to 0$ in probability. Parts (iii)-(iv) follow from part (i) and from Lemma 2.3.

The proof of part (ii) is slightly more subtle because we need to parlay the convergence of relaxed control measures from Lemma 4.4 into a statement about the convergence in $L^2([0,T];I\!R^n)$ of the controls $u^h(X^h(t))$. Consider a sequence of initial conditions $x^h \in B^h$ such that $x^h \to x \in \overline{B}$ as $h \to 0$. Using Lemma 4.4 and the Skorokhod Representation Theorem, we consider a probability space on which $\nu^h \to \nu^0$ w.p.1. Since $\nu^0(du \times dt) = \delta_{u^0(X^0(t))}(du)dt$ and $\nu^h(du \times dt) = \delta_{u^h(X^h(t))}(du)dt$, the w.p.1 convergence $\nu^h \to \nu^0$ implies

$$\begin{aligned} \int_0^T \|u^h(X^h(t)) - u^0(X^0(t))\|^2 dt &= \int_{U\times[0,T]} \|u - u^0(X^0(t))\|^2 \nu^h(du \times dt) \\ &\longrightarrow \int_{U\times[0,T]} \|u - u^0(X^0(t))\|^2 \nu^0(du \times dt) \\ &= \int_0^T \|u^0(X^0(t)) - u^0(X^0(t))\|^2 dt \\ &= 0, \end{aligned}$$

where the limit in the second line holds as $h \to 0$, w.p.1. Thus, switching from the Skorokhod space back to the original random variables, we can conclude that $\|u^h(X^h) - u^0(X^0)\|_T$ converges to zero in probability, for any sequence of initial conditions $x^h \in B^h$ such that $x^h \to x \in \overline{B}$ as $h \to 0$. Now, as in the proof of part (i), this implies the convergence asserted by the lemma. ■

It is useful to identify the suboptimal processes obtained by applying the limit optimal feedback control u^0 in the h-dynamics. For an initial condition in N^h, let $X^{h,0}$ be the process obtained by taking $\underline{u}^h = u^0$ in Section 3, and let $m^{h,0}$ and $Y^{h,0}$ be the corresponding martingale and bounded variation parts indicated by the decomposition (3.2). Finally, define the exit time $\tau_N^{h,0} = \inf[t : X^{h,0}(t) \notin N^h]$ and the exit location $z_N^{h,0} = X^{h,0}(\tau_N^{h,0})$.

Lemma 4.5 *For every $\varepsilon > 0$, there exists $h_0 > 0$ such that for all $0 < h \leq h_0$ and for all initial conditions $x \in B^h$,*

$$(i) \quad P_x\left[||| \, X^{h,0} - X^0 \, |||_T > \varepsilon \right] < \varepsilon$$

$$(ii) \quad P_x\left[|\tau_N^{h,0} - \tau^0| > \varepsilon \right] < \varepsilon$$

$$(iii) \quad P_x\left[\|z_N^{h,0} - z^0\| > \varepsilon \right] < \varepsilon$$

where T is the bound on the exit times from Lemma 2.1.

Proof. For an initial condition $x \in B^h$, let $Z^h(t) = X^{h,0}(t) - X^0(t)$. Then by (3.2) we have

$$Z^h(t) = \int_0^t \Big[u^0(X^{h,0}(s)) - u^0(X^0(s))\Big]\, ds + m^{h,0}(t)$$

holding w.p.1 for any $t < +\infty$. Thus, if K is the uniform Lipschitz constant for u^0, then for any $0 \leq \sigma < +\infty$

$$\|Z^h(t)\| \leq \int_0^t K\|Z^h(s)\|ds + ||| \; m^{h,0}(t) \; |||_\sigma$$

holds for w.p.1 each $0 \leq t \leq \sigma$. We can apply a version of Gronwall's inequality [10, Theorem A.6.4] to get the w.p.1 bound

$$||| \; X^{h,0}(t) - X^0(t) \; |||_\sigma \leq ||| \; m^{h,0}(t) \; |||_\sigma e^{K\sigma}. \tag{4.5}$$

Now, letting $\sigma = T$ in (4.5) and applying Lemma 3.1 with a standard submartingale inequality, we obtain part (i) of the lemma. Parts (ii)-(iii) follow directly from part (i) and from Lemma 2.3. ■

5 Convergence of the Feedback Controls

The main results of this paper are Theorem 5.5 and Corollary 5.6. They state that in the set B_0, the optimal feedback controls $u^h(x)$ for the approximating control problems converge uniformly to $u^0(x)$, the optimal feedback control for the limit problem. Once we establish the analogous convergence of the approximate gradients $D^{h,\pm}V^h(x)$ to $DV^0(x)$, we will be able to use the uniqueness of the optimal control $u^0(x)$ to prove Theorem 5.5. Thus, most of this section is devoted to establishing the following lemma.

Lemma 5.1 *Let $x^h \in B_0^h$ be such that $x^h \to x \in \overline{B_0}$ as $h \to 0$. Then, as $h \to 0$,*

$$D^{h,\pm}V^h(x^h) \to DV^0(x).$$

There are two main steps in the proof of Lemma 5.1. First, we obtain the convergence of $D^{h,\pm}V^h(x)$ to $DV^0(x)$ in a neighborhood of $B \cap \partial G$. Then, we use representations of $V^0(x)$ and $V^h(x)$ in terms of integrals along optimal trajectories to obtain the convergence of the $D^{h,\pm}V^h(x)$ to $DV^0(x)$ on the interior of the smaller region B_0. Our arguments are very similar in spirit to those used in the proof of [14, Lemma 5.5]. It is useful in what follows to define a compact notation for the running cost under a feedback control $u(x)$ by

$$L_u(x) = L(x, u(x)). \tag{5.1}$$

The following lemma establishes a geometric bound on the difference between V^h and V^0 on the set B.

Lemma 5.2 *For any* $m > 0$*, there exists* $h_0 > 0$ *such that*

$$(1-m)V^0(x) \;\le\; V^h(x) \;\le\; (1+m)V^0(x)$$

holds for all $0 < h \le h_0$ *and for all* $x \in B^h$*.*

Proof. We prove this lemma in two steps, first considering the upper bound on $V^h(x)$ and then the lower bound.

Upper Bound:

Let $\mu > m$ and put $W^0 = (1+\mu)V^0$. It follows from the DPE (2.3) that

$$\langle u^{0,+}, D^{h,+}W^0\rangle - \langle u^{0,-}, D^{h,-}W^0\rangle + \tilde{L}_{u^0} = 0 \tag{5.2}$$

holds on B, where the modified cost $\tilde{L}_{u^0}(x)$ is defined on B by

$$\tilde{L}_{u^0} = (1+\mu)L_{u^0} + \langle u^0, DW^0\rangle - \langle u^{0,+}, D^{h,+}W^0\rangle + \langle u^{0,-}, D^{h,-}W^0\rangle.$$

Note that since $L_{u^0}(x) \ge c_0 > 0$ and $W^0(x)$ is smooth, $\tilde{L}_{u^0}(x) \ge L_{u^0}(x)$ for $h > 0$ sufficiently small and for all $x \in B$. Since the generator in (5.2) corresponds to applying the feedback control u^0 in the h-dynamics, we can use a standard verification argument to establish for all $x \in B^h$ the representation

$$W^0(x) = E_x\left[\int_0^{\tau_N^{h,0}} \tilde{L}_{u^0}(X^{h,0})dt + W^0(z_N^{h,0})\right]. \tag{5.3}$$

We use part (ii) of Lemma 4.5 to obtain the uniform integrability of $\tau_N^{h,0}$ needed for the right hand side of (5.3) to be finite.

For $x \in B^h$, we define

$$V^{h,0}(x) = E_x\left[\int_0^{\tau_N^{h,0}} L_{u^0}(X^{h,0})dt + V^h(z_N^{h,0})\right]. \tag{5.4}$$

Since the feedback control u^0 is suboptimal in the control problem with the h-dynamics, it follows from the strong Markov property that $V^h(x) \le$

$V^{h,0}(x)$ for all $x \in B^h$. Thus, it suffices to establish the bound $V^{h,0}(x) \leq (1+m)V^0(x)$.

Let K be the bound on $V^h(x)$ from Lemma 3.3. Then the following series of inequalities holds for all sufficiently small $h > 0$ and for all $x \in B^h$:

$$\begin{aligned} V^{h,0}(x) &\leq W^0(x) + E_x\left[V^h(z_N^{h,0}) - W^0(z_N^{h,0})\right] \\ &\leq (1+\mu)V^0(x) + E_x V^h(z_N^{h,0}) \qquad (5.5) \\ &\leq (1+\mu)V^0(x) + KP_x\left[z_N^{h,0} \in G\right]. \end{aligned}$$

The first line uses the representations (5.3) and (5.4), along with the fact that $\tilde{L}_{u^0}(x) \geq L_{u^0}(x)$ for all $x \in B$; the second line uses the definition of $W^0(x)$ and the non-negativity of $W^0(x)$ for all $x \in G$; and the third line uses the fact that $V^h(x) = 0$ for all $x \in \partial G$.

We now turn our attention to bounding the final term in the last display. Let $\varepsilon > 0$ be equal to $d(B, \partial N \cap G)$, so that $z_N^{h,0} \in G$ implies that $z_N^{h,0}$ is at least a distance of ε away from $z^0 \in B$; see Figure 3 before Lemma 5.3. Choose $0 < \delta < \varepsilon/2$ such that once an optimal trajectory for the limit problem with initial condition $x \in B$ gets to within δ of the boundary ∂G, it can travel no further than distance $\varepsilon/2$ before exiting. The existence of such a $\delta > 0$ is guaranteed by the nontangential exit property for the regions of strong regularity. Finally, let $0 < \eta < \delta < \varepsilon/2$ be chosen so that the conclusions of Lemma 2.3 hold. We obtain the following series of inequalities holding for all $x \in B^h$, each line of which is explained after the display.

$$\begin{aligned} P_x\left[z_N^{h,0} \in G\right] &\leq P_x\left[\|z_N^{h,0} - z^0\| > \varepsilon\right] \\ &\leq P_x\left[\||\, X^{h,0} - X^0 \,\||_{\tau_N^{h,0} \wedge T} \geq \eta\right] \\ &\leq P_x\left[\||\, m^{h,0}(t) \,\||_{\tau_N^{h,0} \wedge T} \geq \eta e^{-K'T}\right] \qquad (5.6) \\ &\leq (\eta e^{-K'T})^{-2} E_x\left[\|m^{h,0}(\tau_N^{h,0} \wedge T)\|^2\right] \\ &\leq hCE_x\tau_N^{h,0} \\ &\leq hc_0^{-1}CV^{h,0}(x) \end{aligned}$$

The first line is a consequence of the fact that $z_N^{h,0} \in G$ can only occur if $\|z_N^{h,0} - z^0\| > \varepsilon$; the second line follows from the choice of δ and η; the third line follows from equation (4.5), with K' equal to the Lipschitz constant for $u^0(x)$; the fourth line is obtained by a standard submartingale inequality; the fifth line follows from Lemma 3.1, with C a composite finite constant; and the last line is a consequence of the definition of $V^{h,0}$ and of the lower bound c_0 on the running cost. We can combine the last lines of (5.5) and (5.6) to obtain the bound

$$(1 - hc_0^{-1}CK)V^{h,0}(x) \le (1+\mu)V^0(x)$$

for sufficiently small $h > 0$ and for all $x \in B^h$. Since $V^h(x) \le V^{h,0}(x)$, we complete the proof of the upper bound by taking $h > 0$ sufficiently small so that $(1+\mu)/(1 - hc_0^{-1}CK) \le 1 + m$.

Lower Bound:

Let $\mu < m$, and this time put $W^0 = (1-\mu)V^0$. It follows from the DPE (2.3) that the relation

$$\langle u^h, DV^0 \rangle + L_{u^h} - \phi^h = 0$$

holds on B^h for some non-negative function ϕ^h. This, in turn, implies that

$$\langle u^{h,+}, D^{h,+}W^0 \rangle - \langle u^{h,-}, D^{h,-}W^0 \rangle + \tilde{L}_{u^h}^h = 0, \tag{5.7}$$

where $\tilde{L}_{u^h}^h(x)$ is defined on B^h by

$$\begin{aligned} \tilde{L}_{u^h}^h &= (1-\mu)L_{u^h} - (1-\mu)\phi^h \\ &\quad + \langle u^h, DW^0 \rangle - \langle u^{h,+}, D^{h,+}W^0 \rangle + \langle u^{h,-}, D^{h,-}W^0 \rangle. \end{aligned}$$

Since $L_{u^h}(x) \ge c_0 > 0$ and $W^0(x)$ is smooth, the non-negativity of $\phi^h(x)$ implies that $L_{u^h}(x) \ge \tilde{L}_{u^h}^h(x)$ for $h > 0$ sufficiently small and for all $x \in B^h$. The generator in (5.7) corresponds to applying the feedback control u^h in the h-dynamics, so we can use a standard verification argument to establish for all $x \in B^h$ the representation

$$W^0(x) = E_x \left[\int_0^{\tau_N^h} \tilde{L}_{u^h}^h(X^h)dt + W^0(z_N^h) \right]. \tag{5.8}$$

We use part (iii) of Lemma 4.1 to obtain the uniform integrability of τ_N^h needed for the right hand side to be finite. The strong Markov property implies the representation,

$$V^h(x) = E_x \left[\int_0^{\tau_N^h} L_{u^h}(X^h) dt + V^h(z_N^h) \right]. \tag{5.9}$$

Thus, the following series of inequalities holds for all sufficiently small $h > 0$ and for all $x \in B^h$:

$$\begin{aligned} V^h(x) &\geq W^0(x) + E_x \left[V^h(z_N^h) - W^0(z_N^h) \right] \\ &\geq (1-\mu)V^0(x) - \sup_{y \in G^h} \left| V^h(y) - V^0(y) \right| P_x \left[z_N^h \notin \partial G \right] \\ &= (1-\mu)V^0(x) - o_h(1) P_x \left[z_N^h \notin \partial G \right] \end{aligned} \tag{5.10}$$

The first line follows from the representations (5.8) and (5.9), along with the fact that $L_{u^h}(x) \geq \tilde{L}^h_{u^h}(x)$ for all $x \in B^h$; the second line uses the definition of $W^0(x)$, the non-negativity of $V^0(x)$ for all $x \in G$, and the fact that $V^h(x) = V^0(x) = 0$ for all $x \in \partial G$; finally, the third line uses part (ii) of Theorem 4.2, and the $o_h(1)$ term converges to zero as $h \to 0$, uniformly for all $x \in B^h$.

Let $\varepsilon > 0$ be equal to $d(B, \partial N \cap G)/2$. Notice that any trajectory with an initial condition $x \in B$ must travel a distance of at least 2ε if it is to exit N at a point which is not in ∂G; see Figure 3. Let $K \geq \sup_{u \in U} \|u\|$ be such that $\varepsilon / K \leq T$. Then, given the semi-martingale decomposition in (3.2), $z_N^h \in \partial G$ will follow if $\tau_N^h < \varepsilon/K \leq T$ and if $\vert\vert\vert\, m^h \,\vert\vert\vert_{\tau_N^h \wedge T} < \varepsilon$. By Chebyshev's inequality and by the lower bound c_0 on the running cost, for each $x \in B^h$ we have

$$\begin{aligned} P_x \left[\tau_N^h \geq \varepsilon/K \right] &\leq (\varepsilon/K)^{-1} E_x \tau_N^h \\ &\leq (\varepsilon c_0 / K)^{-1} V^h(x). \end{aligned}$$

As in (5.6), we can use a standard submartingale inequality and Lemma 3.1 to verify that

$$P_x \left[\vert\vert\vert\, m^h \,\vert\vert\vert_{\tau_N^h \wedge T} \geq \varepsilon \right] \leq h C V^h(x)$$

for all $x \in B^h$, where C is a finite constant. Thus, for a composite constant C', we conclude that

$$P_x \left[z_N^h \notin \partial G \right] \leq C' V^h(x)$$

for all $x \in B^h$. Combining this last bound with (5.10), we obtain

$$(1 + o_h(1))V^h(x) \geq (1 - \mu)V^0(x)$$

for all $h > 0$ sufficiently small and for all $x \in B^h$. By taking $h > 0$ sufficiently small so that $(1 - \mu)/(1 + o_h(1)) \geq 1 - m$ on B^h, we complete the proof of the lower bound. ■

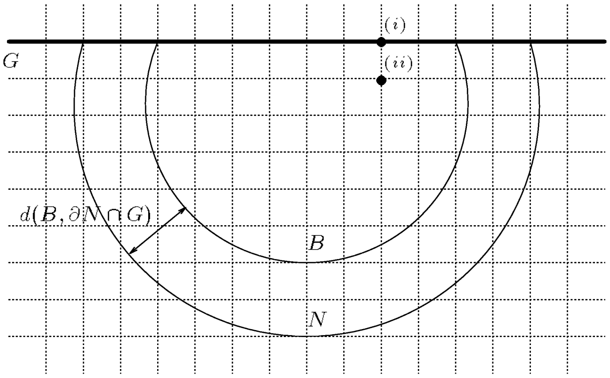

Figure 3: Boundary Points

We are now able to prove that the approximations $D^{h,\pm}V^h(x)$ converge uniformly to the gradient $DV^0(x)$ at appropriate points x near the boundary of B. The two cases in the following lemma are illustrated in Figure 3.

Lemma 5.3 *For* $\varepsilon > 0$ *there exists* $h_0 > 0$ *such that for all* $0 < h \leq h_0$ *and for each* $i = 1, \ldots, n$,

$$|D_i^{h,+}V^h(x) - D_iV^0(x)| \leq \varepsilon \quad (\text{resp., } |D_i^{h,-}V^h(x) - D_iV^0(x)| \leq \varepsilon)$$

for each $x \in I\!\!R^n$ *such that either* **(i)** $x \in \partial G$ *and* $x + he_i \in B^h$ (resp., $x - he_i \in B^h$) , *or* **(ii)** $x \in B^h$ *and* $x + he_i \in \partial G$ (resp., $x - he_i \in \partial G$).

Proof. For simplicity, we treat only case (i) with $x \in \partial G$ and $x + he_i \in B^h$. Given the smoothness of D_iV^0, the other cases follow easily from the same argument. Fix $\varepsilon > 0$ and let K be the uniform Lipschitz constant for $V^0(x)$. Then by Lemma 5.2, there exists $h_0 > 0$ such that

$$\left|\frac{V^h(x + he_i) - V^0(x + he_i)}{V^0(x + he_i)}\right| \leq \frac{\varepsilon}{2}K^{-1} \tag{5.11}$$

for all $0 < h \le h_0$ and for all $x \in I\!R^n$ satisfying condition (i). For such x and h, put $y = x + he_i$. Then, using (5.11) along with the fact that $V^0(x)$ and $V^h(x)$ satisfy zero boundary conditions on ∂G, we obtain

$$\begin{aligned} |D_i^{h,+}V^h(x) - D_i^{h,+}V^0(x)| &= \left|\frac{V^h(y) - V^0(y)}{h}\right| \\ &= \left|\frac{V^0(y)}{h}\right|\left|\frac{V^h(y) - V^0(y)}{V^0(y)}\right| \\ &\le K\left|\frac{V^h(y) - V^0(y)}{V^0(y)}\right| \\ &\le \varepsilon/2. \end{aligned} \tag{5.12}$$

Now let $h_0 > 0$ be sufficiently small so that

$$|D_i^{h,+}V^0(x) - D_iV^0(x)| \le \varepsilon/2$$

for all $0 < h \le h_0$ and $x \in \overline{B}$. Then the result follows from (5.12). ■

The first step in extending the result of Lemma 5.3 to the interior of B_0 is to establish a representation for $DV^0(x)$ in terms of an integral of the gradient in x of the running cost $L(x,u)$ along the optimal trajectories. The proof we give for this representation in the next lemma is fairly simple because it involves only deterministic trajectories. An analogous argument, involving stochastic trajectories, will be used to establish the convergence of $D^{h,\pm}V^h(x)$ to $DV^0(x)$ in the proof of Lemma 5.1. Recall the notation $L_u(x) = L(x, u(x))$ for a feedback control $u(x)$.

Lemma 5.4 *The representation*

$$DV^0(x) = \int_0^{\tau_x^0} DL_{u^0}(X_x^0)dt + DV^0(z_x^0) \tag{5.13}$$

holds for all initial conditions $x \in \overline{B_0}$.

Proof. We fix $x \in \overline{B_0}$ and establish the representation separately for each component $D_iV^0(X)$ of the gradient $DV^0(x)$. Without loss of generality, assume that $x + he_i \in N$ for all sufficiently small $h > 0$. Since $D_i^{h,+}V^0(x) \to$

$D_i V^0(x)$ as $h \to 0$, we can establish the representation by proving separately the upper bound

$$\limsup_{h \to 0} D_i^{h,+} V^0(x) \le \int_0^{\tau_x^0} D_i L_{u^0}(X_x^0) dt + D_i V^0(z_x^0)$$

and the lower bound

$$\liminf_{h \to 0} D_i^{h,+} V^0(x) \ge \int_0^{\tau_x^0} D_i L_{u^0}(X_x^0) dt + D_i V^0(z_x^0).$$

Upper Bound:

Let $\hat{\tau}_h$ be the minimum of τ_x^0 and the first exit time from N of the shifted trajectory $X_x^0(t) + he_i$, and let $\hat{z}_h = X_x^0(\hat{\tau}_h)$. Using the fact that $X_x^0(t) + he_i$ is a suboptimal trajectory for the initial condition $x + he_i$, we obtain the relations

$$\begin{aligned} D_i^{h,+} V^0(x) &\le \frac{1}{h} \left[\int_0^{\hat{\tau}_h} L(X_x^0 + he_i, u^0(X_x^0)) dt + V^0(\hat{z}_h + he_i) \right. \\ &\quad \left. - \int_0^{\hat{\tau}_h} L(X_x^0, u^0(X_x^0)) dt - V^0(\hat{z}_h) \right] \\ &= \int_0^{\hat{\tau}_h} D_i^{h,+} L_{u^0}(X_x^0) dt + D_i^{h,+} V^0(\hat{z}_h). \end{aligned}$$

Lemma 2.3 implies that $|\hat{\tau}_h - \tau_x^0|$ and $\|\hat{z}_h - z_x^0\|$ both converge to zero as $h \to 0$, so we can apply the Lebesgue Dominated Convergence Theorem to obtain the upper bound.

Lower Bound:

This time, let $\hat{\tau}_h$ be the minimum of $\tau_{x+he_i}^0$ and the first exit time from N of the shifted trajectory $X_{x+he_i}^0(t) - he_i$, and let $\hat{z}_h = X_{x+he_i}^0(\hat{\tau}_h)$. Using the fact that $X_{x+he_i}^0(t) - he_i$ is a suboptimal trajectory for the initial condition x, we obtain the relations

$$\begin{aligned} D_i^{h,+}V^0(x) &\geq \frac{1}{h}\left[\int_0^{\hat{\tau}_h} L(X^0_{x+he_i}, u^0(X^0_{x+he_i}))dt + V^0(\hat{z}_h)\right. \\ &\quad \left. - \int_0^{\hat{\tau}_h} L(X^0_{x+he_i} - he_i, u^0(X^0_{x+he_i}))dt - V^0(\hat{z}_h - he_i)\right] \\ &= \int_0^{\hat{\tau}_h} D_i^{h,-}L_{u^0}(X^0_{x+he_i})dt + D_i^{h,-}V^0(\hat{z}_h). \end{aligned}$$

By Lemma 2.4, $\|| X^0_{x+he_i} - X^0_x \||_T$ converges to zero as $h \to 0$. Thus, Lemma 2.3 implies that $|\hat{\tau}_h - \tau^0_x|$ and $\|\hat{z}_h - z^0_x\|$ both converge to zero as $h \to 0$, and we can apply the Lebesgue Dominated Convergence Theorem to obtain the lower bound. ■

We can now use the representation for $DV^0(x)$ obtained in Lemma 5.4 to prove Lemma 5.1. In fact, the proofs are essentially analogous. The primary difference is that the trajectories which arise in the proof of Lemma 5.1 are stochastic, so the analysis of the limits as $h \to 0$ is more involved.

Proof of Lemma 5.1. We give a detailed argument only for the convergence $D_i^{h,+}V^h(x^h) \to D_iV^0(x)$. Fix $x \in \overline{B_0}$, and let $x^h \in B_0^h$ be such that $x^h \to x$ as $h \to 0$. As usual, we prove separately the upper bound

$$\limsup_{h\to 0} D_i^{h,+}V^h(x^h) \leq D_iV^0(x)$$

and the lower bound

$$\liminf_{h\to 0} D_u^{h,+}V^h(x^h) \geq D_iV^0(x).$$

Upper Bound:

Let $\hat{\tau}_h$ be the minimum of $\tau^h_{x^h,N}$ and the first exit time from the interior of N of the shifted trajectory $X^h_{x^h}(t) + he_i$, and let $\hat{z}_h = X^h_{x^h}(\hat{\tau}_h)$. Using the fact that the trajectory $X^h_{x^h}(t) + he_i$ results when the suboptimal feedback control $\tilde{u}^h(\cdot) = u^h(\cdot - he_i)$ is applied in the h-dynamics with initial condition

$x^h + e_i$, we obtain the relations

$$\begin{aligned} D_i^{h,+} V^h(x^h) &\leq \frac{1}{h} E_x^h \left[\int_0^{\tilde{\tau}_h} L_{\tilde{u}^h}(X_{x^h}^h + he_i)dt + V^h(\tilde{z}_h + he_i) \right. \\ &\qquad \left. - \int_0^{\tilde{\tau}_h} L_{u^h}(X_{x^h}^h)dt - V^h(\tilde{z}_h) \right] \\ &= E_{x^h} \left[\int_0^{\hat{\tau}_h} D_i^{h,+} L_{u^h}(X_{x^h}^h)dt + D_i^{h,+} V^h(\hat{z}_h) \right]. \end{aligned}$$

In light of the representation (5.13), this implies that we can establish the upper bound by showing that

$$E_{x^h} \left| \int_0^{\hat{\tau}_h} D_i^{h,+} L(X_{x^h}^h, u^h(X_{x^h}^h))dt - \int_0^{\tau_x^0} D_i L(X_x^0, u^0(X_x^0))dt \right| \tag{5.14}$$

and

$$E_{x^h} \left| D_i^{h,+} V^h(\hat{z}_h) - D_i V^0(z_{x,N}^0) \right| \tag{5.15}$$

both converge to zero as $h \to 0$. Recall from Remark 3.2 and Lemma 3.3 that the pair $(X^h, u^h(X^h))$ take values in a compact set for all initial conditions and for all $h \geq 0$. Thus, the smoothness of L implies that we can use the triangle inequality to bound the quantity in (5.14) by a constant times

$$E_{x^h} \left[\|X_{x^h}^h - X_x^0\|_T + \|u^h(X_{x^h}^h) - u^0(X_x^0)\|_T + |\hat{\tau}_h - \tau_x^0| + hT \right]. \tag{5.16}$$

By combining parts (i) and (ii) of Lemma 4.1 with Lemma 2.4, we can establish that the first two terms in (5.16) converge to zero as $h \to 0$. Uniform integrability of the $\hat{\tau}_h$ follows from the strong Markov property and from the fact that Lemmas 2.3 and 2.4 together imply that $\hat{\tau}_h \leq T$ with positive probability, uniformly in h. Thus, since $\hat{\tau}_h \to \tau_x^0$ in probability, the third term in (5.16) converges to zero as $h \to 0$, and this implies that the expression in (5.14) converges to zero as $h \to 0$.

Applying the triangle inequality to (5.15), we find that it is bounded by

$$E_{x^h} \left[|D_i^{h,+} V^h(\hat{z}_h) - D_i V^0(\hat{z}_h)| + |D_i V^0(\hat{z}_h) - D_i V^0(z_x^0)| \right]. \tag{5.17}$$

Combining part (i) of Lemma 4.1 with Lemmas 2.3 and 2.4, we conclude that $\hat{z}_h$ converges to z_x^0 in probability. The continuity result in Lemma 2.4 and the fact that B is a region of strong regularity imply that the set $\{z_y^0 : y \in \overline{B_0}\}$ is compactly contained in B, so the probability of $\hat{z}^h$ satisfying the conditions of Lemma 5.3 increases to one as $h \to 0$. Thus, we can use Lemma 5.3 and the smoothness of $D_i V^0$ to conclude that each of the terms in (5.17) converges to zero as $h \to 0$. That, in turn, implies that (5.14) converges to zero as $h \to 0$ and completes the proof of the upper bound.

Lower Bound:

This time, we let $\hat{\tau}_h$ be the minimum of $\tau^h_{x+he_i}$ and the first exit time from N of the shifted trajectory $X^h_{x+he_i}(t) - he_i$, and let $\hat{z}_h = X^h_{x+he_i}(\hat{\tau}_h)$. As in the proof of the upper bound, we obtain

$$D_i^{h,+} V^h(x^h) \quad \geq \quad E_{x^h}\left[\int_0^{\hat{\tau}_h} D_i^{h,-} L_{u^h}(X^h_{x^h+he_i})dt + D_i^{h,-} V^h(\hat{z}_h)\right].$$

See also the analogous relation which appears in the proof of the lower bound for Lemma 5.4. Just as in the proof of the upper bound, we show that the right hand side of the above expression converges to $D_i V^0(x)$ as $h \to 0$. Notice that we need $x^h + he_1 \in B^h$ in order to apply Lemma 4.1. Since $x^h \in B_0$, this condition is satisfied for all sufficiently small $h > 0$. ■

Theorem 5.5 *Let $x^h \in B_0^h$ be such that $x^h \to x \in \overline{B_0}$ as $h \to 0$. Then, as $h \to 0$,*

$$u^h(x^h) \to u^0(x).$$

Proof. For each $u \in U$, we define

$$F^0(u) = \langle u, DV^0(x)\rangle + L(x,u)$$

and

$$F^h(u) = \langle u^+, D^{h,+}V^h(x^h)\rangle - \langle u^-, D^{h,-}V^h(x^h)\rangle + L(x^h, u).$$

Recall from Lemma 3.3 that the optimal controls u^0 and u^h take values in the compact set U. Thus, there exists $\tilde{u} \in U$ such that in a subsequence, $u^h(x^h) \to \tilde{u}$ as $h \to 0$. It suffices to show that $\tilde{u} = u^0(x)$. Since x is in a region of strong regularity, it follows from the DPE (2.3) that the unique minimizer of $F^0(u)$ is given by $u^0(x)$. Similarly, the DPE (3.5) implies that $u^h(x^h)$ is a minimizer of $F^h(u)$. Also, notice that $F^0(u)$ is a continuous function of u and that Lemma 5.1 implies that $F^h(u)$ converges to $F^0(u)$ for each $u \in U$. Thus, we obtain the following relations:

$$F^0(u^0(x)) \;=\; \lim_{h\to 0} F^h(u^0(x)) \;\geq\; \lim_{h\to 0} F^h(u^h(x^h)) \;=\; F^0(\tilde{u}).$$

Since $u^0(x)$ is the unique minimizer of F^0, it follows that $\tilde{u} = u^0(x)$. That completes the proof of the theorem. ■

Corollary 5.6 *For any* $\varepsilon > 0$*, there exist* $h_0 > 0$ *such that*

$$\|u^h(x) - u^0(x)\| < \varepsilon$$

for all $0 < h \leq h_0$ *and for all* $x \in B_0^h$.

Proof. Since $u^0(x)$ is uniformly continuous on $\overline{B_0}$, the result follows from Theorem 5.5 and a standard argument by contradiction. ■

6 Computational Examples

In this section, we present examples of approximations obtained by numerically solving the DPE (3.5) for the Markov chain optimal control problem. The fixed point of that equation is taken as the approximation to the value function $V^0(x)$, and the infimizing feedback control is taken as an approximate feedback control for the limit problem. The solution of (3.5) is obtained by using either Jacobi or Gauss-Seidel iteration in the equivalent equation (3.6), and we note that the infima at each step can be evaluated analytically [5]. When the Gauss-Seidel method is used, we observe in our examples that the number of iterations required to find the fixed point is essentially independent of the parameter h.

Example 1: Our first example is a minimum escape time problem on the unit square $G = [-1,1] \times [-1,1]$ in $\mathbb{R}^2$, with running cost

$$L(x,u) = \frac{1}{4}\|u\|^2 + 1.$$

The domain can be decomposed into four regions of strong regularity on which the value function is smooth and on which there is a smooth optimal feedback control. This decomposition and the optimal values are indicated in Figure 4a. Arrows indicate the direction of the optimal velocity field. Figure 4b displays the values of the first component of the feedback control $u^h(x)$ for $h = 0.05$. We see that the discontinuities in the optimal control are resolved very sharply by our approximation scheme. In Table 1, we indicate errors in the approximations to the controls. The L^1 errors reported there are for the entire domain, while the L^∞ errors are for points

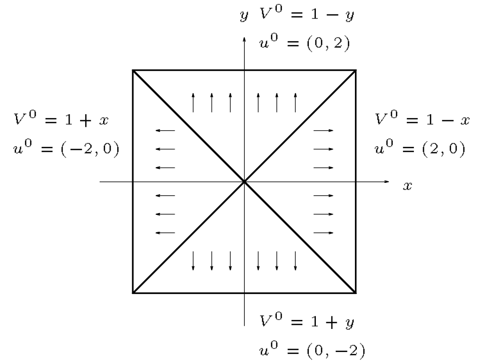

Figure 4a: Exact Solution

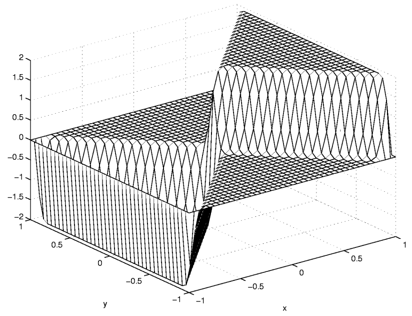

Figure 4b: Approximate Control

inside the regions of strong regularity and a distance of at least 0.1 from the discontinuities. We also indicate the number of iterations required for the Gauss-Seidel method to achieve a residual of less than 10^{-8}, our standard tolerance. We see that the errors in the optimal controls become machine

	$n=2$			$n=3$		
h	Iter	L^1	L^∞ RSR	Iter	L^1	L^∞ RSR
0.2	8	1.74 e − 00	5.91 e − 01	9	4.36 e − 00	5.91 e − 01
0.1	8	1.01 e − 00	8.77 e − 02	17	2.77 e − 00	8.77 e − 02
0.05	8	5.43 e − 01	1.92 e − 03	17	1.55 e − 00	1.92 e − 03
0.025	8	2.80 e − 01	2.31 e − 13	17	8.22 e − 01	3.94 e − 13

Table 1: Escape Time Problem Errors

zero on the regions of strong regularity, in part a consequence of the fact that the value function is linear in those regions. Additionally, the optimal controls appear to converge in L^1 on the entire domain. Without detailed assumptions about the structure of the regions of strong regularity, our results do not necessarily predict that type of convergence. However, since we know for the present problem that the complement of the regions of strong regularity has Lebesgue measure zero, convergence in L^1 on the entire domain is, in fact, expected. Finally, similar error values are indicated in the second part of Table 1 for the escape time problem on the unit cube in $\mathbb{R}^3$.

Example 2: Our next example involves a value function which is obtained by perturbing the value function for the escape time problem in $I\!R^2$. We take care to modify the value function and the cost structure in such a way that we obtain a new problem with smooth data and with a solution that can be evaluated analytically. To that end, we introduce the C^∞ double bump function defined by

$$\chi(\xi) = \begin{cases} e^{-\lambda((\xi-m)^2-\sigma^2)^{-2}} & \xi \in [m-\sigma, m+\sigma] \\ e^{-\lambda((-\xi-m)^2-\sigma^2)^{-2}} & \xi \in [-m-\sigma, -m+\sigma] \\ 0 & \text{otherwise} \end{cases}$$

where we use the parameter values $m = 0.7$, $\sigma = 0.5$, and $\lambda = 0.07$. Now we define a mollifier by

$$\Phi(x,y) = \chi(x+y)\chi(x-y)$$

for all (x,y) in the unit square, and then define the value function $V^0(x,y)$ by multiplying the value function for the escape time problem by $1+\Phi(x,y)$. The resulting function has the same regions of strong regularity as indicated in Figure 4a, and it maintains the linear structure in a neighborhood of the discontinuities. In a similar spirit, we define

$$a(x,y) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 3\sin(2\pi x)^2 \begin{bmatrix} 2 & 5 \\ 5 & 18 \end{bmatrix} \Phi(x,y)$$

and

$$b(x,y) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + 5 \begin{bmatrix} x \\ y\sin((x^2+y^2)^{1/2} - 1/2) \end{bmatrix} \Phi(x,y),$$

so that $a(x,y)$ is the identity and $b(x,y)$ is the zero vector in a neighborhood of the discontinuities. Now, we define $c(x,y)$ on the regions of strong regularity by

$$c(x,y) = (1/2)\Big\langle DV^0(x,y), a(x,y)DV^0(x,y)\Big\rangle - \Big\langle b(x,y), DV^0(x,y)\Big\rangle.$$

Our use of a mollifier in defining all of the relevant functions ensures that the cost function $c(x,y)$ extends smoothly to $c(x,y) = 1$ at the discontinuities, and it turns out that $V^0(x)$ solves the limit control problem for the indicated cost structure.

The optimal trajectories for this problem are indicated in Figure 5a, while trajectories computed using the approximate optimal controls with $h = 0.025$ are shown in Figure 5b. Clearly, the controls computed with

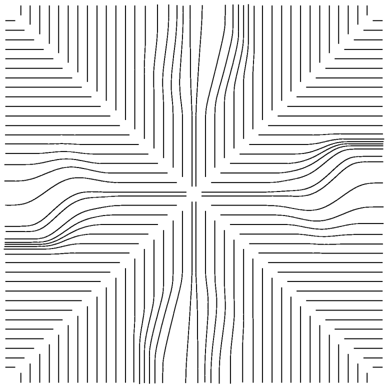

Figure 5a: Characteristics

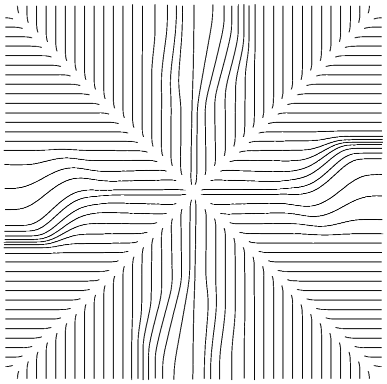

Figure 5b: Approximate Characteristics

h	Iter	L^1	L^∞ RSR
0.1	10	$5.83 \text{ e} - 01$	$2.66 \text{ e} - 01$
0.05	12	$3.24 \text{ e} - 01$	$1.59 \text{ e} - 01$
0.025	12	$1.72 \text{ e} - 01$	$8.94 \text{ e} - 02$
0.0125	12	$8.93 \text{ e} - 02$	$4.84 \text{ e} - 02$
0.00625	12	$4.55 \text{ e} - 02$	$2.54 \text{ e} - 02$

Table 2: Perturbed Escape Time Problem Errors

our algorithm yield an excellent approximation to the optimal trajectories. In Table 2, we exhibit error values for the approximations to the optimal control, with the L^1 errors being on the entire domain and the L^∞ errors being for points a distance of at least 0.1 from the discontinuities. Evidently, both measures of the error are approximately proportional to h, and it is also worth noting that the number of iterations required for the Gauss-Seidel procedure to converge to the fixed point is essentially constant. In Figure 6, we display an approximation to the first component of the optimal control with $h = 0.05$ and the errors in the approximation to the control for $h = 0.05$. The discontinuities are resolved very sharply, and we can see that the errors are uniformly small within the regions of strong regularity

Example 3: Our final example is an application to the problem of finding geodesics on a surface, suggested in [20]. Given a surface $z(x, y)$ on the unit square in $\mathbb{R}^2$, the problem is to find the shortest path along the surface

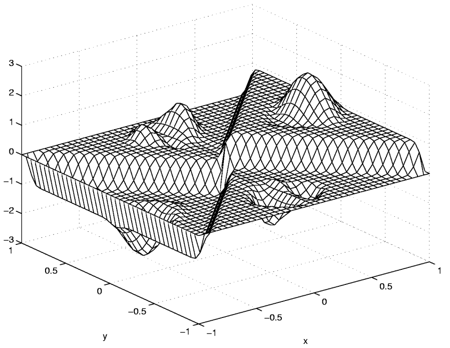

Figure 6a: Approximate Control

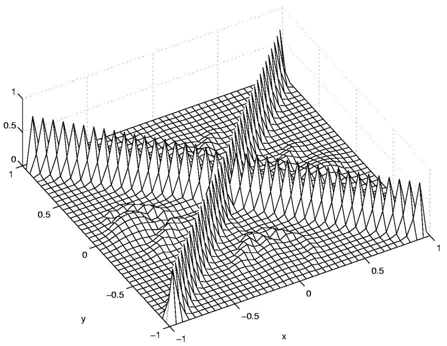

Figure 6b: Control Error

from a given point to the boundary. It is shown in [20, Section 16.5] that the solution to this problem can be obtained from our optimal control problem with running cost specified by

$$a(x,y) = \begin{bmatrix} 1+z_y^2 & z_x z_y \\ z_x z_y & 1+z_x^2 \end{bmatrix}, \qquad b(x,y) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

and

$$c(x,y) = (1/2)(1+z_x^2+z_y^2).$$

Geodesics are obtained by following the optimal trajectories from points on the interior of the unit square to the boundary. In Figure 7a, we show sev-

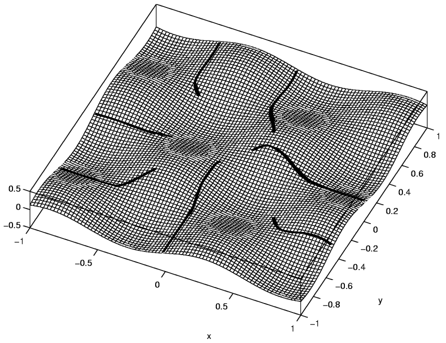

Figure 7a: Geodesics

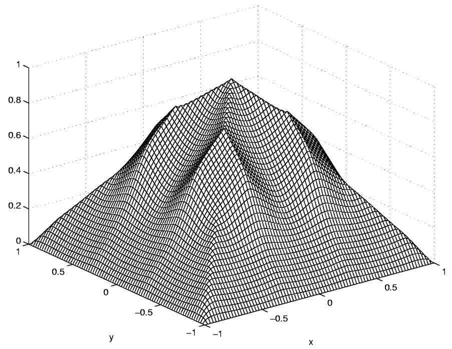

Figure 7b: Value Function

eral approximate geodesics computed using our algorithm for the sinusoidal surface

$$z(x, y) = (1/4)\sin(\frac{7\pi}{4}x)\sin(\frac{7\pi}{4}y).$$

The approximate controls are computed on a grid with spacing $h = 0.025$, and the trajectories are integrated by a simple Euler method with linear interpolation. In Figure 7b, we show the value function computed with $h = 0.025$ for the corresponding control problem, illustrating the fairly complex structure of the regions of strong regularity. Since we do not know the exact solution for this problem, it is not possible for us to present a numerical measure of accuracy for the approximate geodesics in Figure 7a. However, Theorem 5.5 guarantees that for initial conditions in a region of strong regularity, the approximations will converge to the correct geodesics as $h \rightarrow 0$. Being that a more refined grid does not result in discernible changes to the paths indicated in Figure 7a, we conclude that these are, in fact, good approximations to the exact geodesic curves.

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