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e-companion
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Electronic Companion—"Performance of Portfolios Optimized with Estimation Error" by Andrew F. Siegel and Artemiza Woodgate, *Management Science* 2007, 53(6) 1005–1015.

Online Appendix. Proofs

This appendix briefly reviews the notation and definitions and establishes basic lemmas before proving the results of the paper. Basic second-order expectations are given in Lemma 1, Lemma 2 gives the expansion of a matrix inverse, Lemma 3 shows first- and second-order expansions of $(\mathbf{1}\ \hat{\mu})$ and $\widehat{V}^{-1}$, Lemma 4 shows expansions of $\widehat{B}$, Lemma 5 shows expansions of $\widehat{w}$, Lemma 6 derives matrix traces, and Lemma 7 derives the expectations that are needed to derive the adjusted performance measures and their properties.

We define the first-order statistical differential terms δ and ε so that $\hat{\mu} = \mu + \delta$ and $\widehat{V} = V + \varepsilon$, while B is defined as $B = [(\mathbf{1}\ \mu)'V^{-1}(\mathbf{1}\ \mu)]^{-1}$ so that $w = V^{-1}(\mathbf{1}\ \mu)B\binom{1}{\mu_0}$, while B is estimated as $\widehat{B} = [(\mathbf{1}\ \hat{\mu})'\widehat{V}^{-1}(\mathbf{1}\ \hat{\mu})]^{-1}$. We also define the scalar $\xi = (0\ 1)B(1\ \mu_0)' = B_{22}(\mu_0 - \mu_*)$, where $\mu_* = -B_{12}/B_{22}$. We will let $S_i(f)$ denote the term of order i in the Taylor series expansion of f, while $S_i(f, \delta)$ denotes the term of order i including only δ (i.e., setting $\varepsilon = 0$), and similarly $S_i(f, \varepsilon)$ is obtained by including only ε terms with $\delta = 0$. We will use $E_\Delta[f(\hat{\mu}, \widehat{V})]$ to denote the expectation of the second-order expansion of f at (μ, V) with respect to δ and ε so that

$$\begin{aligned} E_\Delta[f(\hat{\mu}, \widehat{V})] &= f(\mu, V) + E\{S_2[f(\hat{\mu}, \widehat{V})]\} \\ &= f(\mu, V) + E\{S_2[f(\hat{\mu}, \widehat{V}), \delta]\} + E\{S_2[f(\hat{\mu}, \widehat{V}), \varepsilon]\} \end{aligned} \quad \text{(EC1)}$$

because any second-order term that includes both δ and ε will have expectation zero.

Lemma 1. *For any symmetric $n \times n$ matrix Q, $E(\varepsilon Q \varepsilon) = [VQV + V\mathrm{tr}(QV)]/(T-1)$ and $E(\delta' Q \delta) = \mathrm{tr}(QV)/T$.*

Proof. The expectation involving ε follows from Theorem 3.1(iii) of Haff (1979), while the expectation involving δ may be found, for example, Seber (1984, p. 14). □

Lemma 2. *The first- and second-order expansion terms of the inverse of the matrix $Q + \theta$, where Q and θ are $n \times n$ matrices with both Q and $Q + \theta$ invertible, are given by*

$$S_1[(Q+\theta)^{-1}, \theta] = -Q^{-1}\theta Q^{-1}, \quad \text{(EC2)}$$

$$S_2[(Q+\theta)^{-1}, \theta] = Q^{-1}\theta Q^{-1}\theta Q^{-1}. \quad \text{(EC3)}$$

Proof. Begin by observing directly that the identity matrix may be written as

$$I_n = (Q+\theta)(Q^{-1} - Q^{-1}\theta Q^{-1} + Q^{-1}\theta Q^{-1}\theta Q^{-1}) - \theta Q^{-1}\theta Q^{-1}\theta Q^{-1}. \quad \text{(EC4)}$$

Next, premultiply both sides by $(Q+\theta)^{-1}$ to obtain

$$\begin{aligned} (Q+\theta)^{-1} &= Q^{-1} - Q^{-1}\theta Q^{-1} + Q^{-1}\theta Q^{-1}\theta Q^{-1} - (Q+\theta)^{-1}\theta Q^{-1}\theta Q^{-1}\theta Q^{-1} \\ &= Q^{-1} - Q^{-1}\theta Q^{-1} + Q^{-1}\theta Q^{-1}\theta Q^{-1} + O(\|\theta\|^3), \end{aligned} \quad \text{(EC5)}$$

from which we can identify the first- and second-order terms. □

LEMMA 3. *First- and second-order expansions of* $(\mathbf{1}\ \ \hat{\mu})$ *and* $\widehat{V}^{-1}$ *are as follows*:

$$S_1[(\mathbf{1}\quad \hat{\mu}),\delta]=\delta(0\quad 1), \tag{EC6}$$

$$S_2[(\mathbf{1}\quad \hat{\mu}),\delta]=(\mathbf{0}\quad \mathbf{0}), \tag{EC7}$$

$$S_1(\widehat{V}^{-1},\varepsilon)=-V^{-1}\varepsilon V^{-1}, \tag{EC8}$$

$$S_2(\widehat{V}^{-1},\varepsilon)=V^{-1}\varepsilon V^{-1}\varepsilon V^{-1}, \tag{EC9}$$

$$S_1[(\mathbf{1}\quad \hat{\mu}),\varepsilon]=S_2[(\mathbf{1}\quad \hat{\mu}),\varepsilon]=(\mathbf{0}\quad \mathbf{0}),\qquad \textit{and}\qquad S_1(\widehat{V}^{-1},\delta)=S_2(\widehat{V}^{-1},\delta)=\mathbf{0}. \tag{EC10}$$

PROOF. Equations (EC6) and (EC7) follow directly from the definition $\hat{\mu}=\mu+\delta$, while (EC8) and (EC9) follow from Lemma 2 using $\widehat{V}=V+\varepsilon$. Equation (EC10) follows because $\hat{\mu}$ does not depend on ε and $\widehat{V}$ does not depend on δ. □

LEMMA 4. *First- and second-order expansions of* $\widehat{B}$ *with respect to* δ *and* ε *are as follows*:

$$S_1(\widehat{B},\delta)=-B\left[\begin{pmatrix}0\\1\end{pmatrix}\delta' V^{-1}(\mathbf{1}\quad \mu)+(\mathbf{1}\quad \mu)'V^{-1}\delta(0\quad 1)\right]B, \tag{EC11}$$

$$\begin{aligned}S_2(\widehat{B},\delta)={}&\delta'[V^{-1}(\mathbf{1}\quad \mu)B(\mathbf{1}\quad \mu)'V^{-1}-V^{-1}]\delta B\begin{pmatrix}0\\1\end{pmatrix}(0\quad 1)B+B_{22}B(\mathbf{1}\quad \mu)'V^{-1}\delta\delta'V^{-1}(\mathbf{1}\quad \mu)B\\&+B\begin{pmatrix}0\\1\end{pmatrix}(0\quad 1)B(\mathbf{1}\quad \mu)'V^{-1}\delta\delta'V^{-1}(\mathbf{1}\quad \mu)B+B(\mathbf{1}\quad \mu)'V^{-1}\delta\delta'V^{-1}(\mathbf{1}\quad \mu)B\begin{pmatrix}0\\1\end{pmatrix}(0\quad 1)B,\end{aligned} \tag{EC12}$$

$$S_1(\widehat{B},\varepsilon)=B(\mathbf{1}\quad \mu)'V^{-1}\varepsilon V^{-1}(\mathbf{1}\quad \mu)B, \tag{EC13}$$

$$S_2(\widehat{B},\varepsilon)=B(\mathbf{1}\quad \mu)'V^{-1}\varepsilon[V^{-1}(\mathbf{1}\quad \mu)B(\mathbf{1}\quad \mu)'V^{-1}-V^{-1}]\varepsilon V^{-1}(\mathbf{1}\quad \mu)B. \tag{EC14}$$

PROOF. Equations (EC11) and (EC12) follow from identifying first- and second-order terms in the expansion of $\widehat{B}$ with respect to δ, using Lemma 2 while setting $\varepsilon=0$ (because we are expanding with respect to δ only), as follows:

$$\begin{aligned}&[(\mathbf{1}\quad \hat{\mu})'V^{-1}(\mathbf{1}\quad \hat{\mu})]^{-1}\\&\quad=\left\{\left[(\mathbf{1}\quad \mu)'+\begin{pmatrix}0\\1\end{pmatrix}\delta'\right]V^{-1}[(\mathbf{1}\quad \mu)+\delta(0\quad 1)]\right\}^{-1}\\&\quad=\left\{B^{-1}+\begin{pmatrix}0\\1\end{pmatrix}\delta'V^{-1}(\mathbf{1}\quad \mu)+(\mathbf{1}\quad \mu)'V^{-1}\delta(0\quad 1)+\begin{pmatrix}0\\1\end{pmatrix}\delta'V^{-1}\delta(0\quad 1)\right\}^{-1}\\&\quad\cong B-B\left\{\begin{pmatrix}0\\1\end{pmatrix}\delta'V^{-1}(\mathbf{1}\quad \mu)+(\mathbf{1}\quad \mu)'V^{-1}\delta(0\quad 1)+\begin{pmatrix}0\\1\end{pmatrix}\delta'V^{-1}\delta(0\quad 1)\right\}B\\&\qquad+B\left\{\begin{pmatrix}0\\1\end{pmatrix}\delta'V^{-1}(\mathbf{1}\quad \mu)+(\mathbf{1}\quad \mu)'V^{-1}\delta(0\quad 1)\right\}B\left\{\begin{pmatrix}0\\1\end{pmatrix}\delta'V^{-1}(\mathbf{1}\quad \mu)+(\mathbf{1}\quad \mu)'V^{-1}\delta(0\quad 1)\right\}B.\end{aligned} \tag{EC15}$$

By identifying, moving, and transposing selected embedded scalars, the second-order term may be re-expressed as

$$\begin{aligned}S_2(\widehat{B},\delta)={}&-B\begin{pmatrix}0\\1\end{pmatrix}(\delta'V^{-1}\delta)(0\quad 1)B+B\begin{pmatrix}0\\1\end{pmatrix}\left[\delta'V^{-1}(\mathbf{1}\quad \mu)B\begin{pmatrix}0\\1\end{pmatrix}\right]\delta'V^{-1}(\mathbf{1}\quad \mu)B\\&+B\begin{pmatrix}0\\1\end{pmatrix}[\delta'V^{-1}(\mathbf{1}\quad \mu)B(\mathbf{1}\quad \mu)'V^{-1}\delta](0\quad 1)B+B(\mathbf{1}\quad \mu)'V^{-1}\delta\left[(0\quad 1)B\begin{pmatrix}0\\1\end{pmatrix}\right]\delta'V^{-1}(\mathbf{1}\quad \mu)B\\&+B(\mathbf{1}\quad \mu)'V^{-1}\delta[(0\quad 1)B(\mathbf{1}\quad \mu)'V^{-1}\delta](0\quad 1)B\\={}&\delta'[V^{-1}(\mathbf{1}\quad \mu)B(\mathbf{1}\quad \mu)'V^{-1}-V^{-1}]\delta B\begin{pmatrix}0\\1\end{pmatrix}(0\quad 1)B+B_{22}B(\mathbf{1}\quad \mu)'V^{-1}\delta\delta'V^{-1}(\mathbf{1}\quad \mu)B\\&+B\begin{pmatrix}0\\1\end{pmatrix}(0\quad 1)B(\mathbf{1}\quad \mu)'V^{-1}\delta\delta'V^{-1}(\mathbf{1}\quad \mu)B+B(\mathbf{1}\quad \mu)'V^{-1}\delta\delta'V^{-1}(\mathbf{1}\quad \mu)B\begin{pmatrix}0\\1\end{pmatrix}(0\quad 1)B.\end{aligned} \tag{EC16}$$

Equations (EC13) and (EC14) follow from identifying first- and second-order terms in the expansion of $\widehat{B}$ with respect to ε, using Lemma 2 twice while setting $\delta = 0$, as follows:

$$\begin{aligned}[(\mathbf{1}\quad\mu)'\widehat{V}^{-1}(\mathbf{1}\quad\mu)]^{-1} &= [(\mathbf{1}\quad\mu)'(V+\varepsilon)^{-1}(\mathbf{1}\quad\mu)]^{-1}\\ &\cong [(\mathbf{1}\quad\mu)'(V^{-1}-V^{-1}\varepsilon V^{-1}+V^{-1}\varepsilon V^{-1}\varepsilon V^{-1})(\mathbf{1}\quad\mu)]^{-1}\\ &= [B^{-1}-(\mathbf{1}\quad\mu)'V^{-1}\varepsilon V^{-1}(\mathbf{1}\quad\mu)+(\mathbf{1}\quad\mu)'V^{-1}\varepsilon V^{-1}\varepsilon V^{-1}(\mathbf{1}\quad\mu)]^{-1}\\ &\cong B-B[-(\mathbf{1}\quad\mu)'V^{-1}\varepsilon V^{-1}(\mathbf{1}\quad\mu)+(\mathbf{1}\quad\mu)'V^{-1}\varepsilon V^{-1}\varepsilon V^{-1}(\mathbf{1}\quad\mu)]B\\ &\quad +B[-(\mathbf{1}\quad\mu)'V^{-1}\varepsilon V^{-1}(\mathbf{1}\quad\mu)]B[-(\mathbf{1}\quad\mu)'V^{-1}\varepsilon V^{-1}(\mathbf{1}\quad\mu)]B. \quad\square \qquad \text{(EC17)}\end{aligned}$$

Lemma 5. *First-order expansions of $\widehat{w}$ with respect to δ and ε are as follows:*

$$S_1(\widehat{w},\delta) = \left[\xi V^{-1} - V^{-1}(\mathbf{1}\quad\mu)B\begin{pmatrix}0\\1\end{pmatrix}w' - \xi V^{-1}(\mathbf{1}\quad\mu)B(\mathbf{1}\quad\mu)'V^{-1}\right]\delta, \qquad \text{(EC18)}$$

$$S_1(\widehat{w},\varepsilon) = [V^{-1}(\mathbf{1}\quad\mu)B(\mathbf{1}\quad\mu)'V^{-1} - V^{-1}]\varepsilon w. \qquad \text{(EC19)}$$

Proof. Note that $\widehat{w} = \widehat{V}^{-1}(\mathbf{1}\ \hat{\mu})\widehat{B}\binom{1}{\mu_0}$. Expanding with respect to δ, we may set $\varepsilon = 0$ using (EC6) and (EC11) to find

$$\begin{aligned}S_1(\widehat{w},\delta) &= V^{-1}S_1[(\mathbf{1}\quad\hat{\mu})\widehat{B},\delta]\begin{pmatrix}1\\\mu_0\end{pmatrix} = V^{-1}\{S_1[(\mathbf{1}\quad\hat{\mu}),\delta]B + (\mathbf{1}\quad\mu)S_1(\widehat{B},\delta)\}\begin{pmatrix}1\\\mu_0\end{pmatrix}\\ &= V^{-1}\left\{\delta(0\quad 1)B - (\mathbf{1}\quad\mu)B\left[\begin{pmatrix}0\\1\end{pmatrix}\delta'V^{-1}(\mathbf{1}\quad\mu) + (\mathbf{1}\quad\mu)'V^{-1}\delta(0\quad 1)\right]B\right\}\begin{pmatrix}1\\\mu_0\end{pmatrix}\\ &= \xi V^{-1}\delta - V^{-1}(\mathbf{1}\quad\mu)B\begin{pmatrix}0\\1\end{pmatrix}\left[\delta'V^{-1}(\mathbf{1}\quad\mu)B\begin{pmatrix}1\\\mu_0\end{pmatrix}\right] - \xi V^{-1}(\mathbf{1}\quad\mu)B(\mathbf{1}\quad\mu)'V^{-1}\delta. \qquad \text{(EC20)}\end{aligned}$$

Transposing an embedded scalar and factoring establishes (EC18). To prove (EC19), expand with respect to ε while setting $\delta = 0$, so that $\hat{\mu} = \mu$, using (EC8) and (EC13) to find

$$\begin{aligned}S_1(\widehat{w},\varepsilon) &= S_1[\widehat{V}^{-1}(\mathbf{1}\quad\mu)\widehat{B},\varepsilon]\begin{pmatrix}1\\\mu_0\end{pmatrix} = S_1(\widehat{V}^{-1},\varepsilon)(\mathbf{1}\quad\mu)B\begin{pmatrix}1\\\mu_0\end{pmatrix} + V^{-1}(\mathbf{1}\quad\mu)S_1(\widehat{B},\varepsilon)\begin{pmatrix}1\\\mu_0\end{pmatrix}\\ &= -V^{-1}\varepsilon V^{-1}(\mathbf{1}\quad\mu)B\begin{pmatrix}1\\\mu_0\end{pmatrix} + V^{-1}(\mathbf{1}\quad\mu)B(\mathbf{1}\quad\mu)'V^{-1}\varepsilon V^{-1}(\mathbf{1}\quad\mu)B\begin{pmatrix}1\\\mu_0\end{pmatrix}, \qquad \text{(EC21)}\end{aligned}$$

which simplifies to complete the proof. □

Lemma 6. *Some useful traces are given by*

$$\mathrm{tr}[V^{-1}(\mathbf{1}\quad\mu)B(\mathbf{1}\quad\mu)'] = 2 \qquad \textit{and} \qquad \text{(EC22)}$$

$$\mathrm{tr}\left[V^{-1}(\mathbf{1}\quad\mu)B\begin{pmatrix}0\\1\end{pmatrix}w'V\right] = B_{22}(\mu_0 - \mu_*). \qquad \text{(EC23)}$$

Proof. Equation (EC22) follows from commutativity of matrices within the trace operator and the definition of B as follows: $\mathrm{tr}[V^{-1}(\mathbf{1}\ \mu)B(\mathbf{1}\ \mu)'] = \mathrm{tr}[(\mathbf{1}\ \mu)'V^{-1}(\mathbf{1}\ \mu)B] = \mathrm{tr}[I_2] = 2$. Equation (EC23) follows for similar reasons as follows:

$$\mathrm{tr}\left[V^{-1}(\mathbf{1}\quad\mu)B\begin{pmatrix}0\\1\end{pmatrix}(w'V)\right] = \mathrm{tr}\left[(w'V)V^{-1}(\mathbf{1}\quad\mu)B\begin{pmatrix}0\\1\end{pmatrix}\right] = \mathrm{tr}\left[(1\quad\mu_0)B\begin{pmatrix}0\\1\end{pmatrix}\right] = \xi = B_{22}(\mu_0-\mu_*). \quad\square \qquad \text{(EC24)}$$

Lemma 7. *Some useful expectations may be computed as follows:*

$$E[S_2(\widehat{B},\varepsilon)] = -\frac{n-2}{T-1}B, \qquad \text{(EC25)}$$

$$E[S_2(\widehat{B},\delta)] = \frac{1}{T}B_{22}B - \frac{n-4}{T}B\begin{pmatrix}0\\1\end{pmatrix}(0\quad 1)B, \qquad \text{(EC26)}$$

$$E[S_2(\widehat{\sigma}_0^2)] = \frac{B_{22}}{T}\sigma_0^2 - \frac{n-4}{T}B_{22}^2(\mu_0-\mu_*)^2 - \frac{n-2}{T-1}\sigma_0^2, \tag{EC27}$$

$$E[S_2(\widehat{w}'\varepsilon\widehat{w})] = -\frac{2(n-2)}{T-1}\sigma_0^2, \tag{EC28}$$

$$E[\delta' S_1(\widehat{w})] = \frac{n-3}{T}B_{22}(\mu_0-\mu_*), \tag{EC29}$$

$$E[S_2(\widehat{w}'V\widehat{w})] = \frac{B_{22}}{T}\sigma_0^2 - \frac{n-4}{T}B_{22}^2(\mu_0-\mu_*)^2 + \frac{n-2}{T-1}\sigma_0^2, \tag{EC30}$$

$$E[S_2(\widehat{w}'\mu\mu'\widehat{w})] = \frac{\sigma_0^2}{T} - 2\mu_0\frac{n-3}{T}B_{22}(\mu_0-\mu_*). \tag{EC31}$$

Proof. To prove Equation (EC25), use Lemma 1 to evaluate the expectation of (EC14), simplify while recognizing that $(\mathbf{1}\ \ \mu)'V^{-1}(\mathbf{1}\ \ \mu) = B^{-1}$, and using (EC22) to evaluate a trace as follows:

$$\begin{aligned} E[S_2(\widehat{B},\varepsilon)] &= B(\mathbf{1}\ \ \mu)'V^{-1}E\{\varepsilon[V^{-1}(\mathbf{1}\ \ \mu)B(\mathbf{1}\ \ \mu)'V^{-1} - V^{-1}]\varepsilon\}V^{-1}(\mathbf{1}\ \ \mu)B \\ &= \frac{1}{T-1}B(\mathbf{1}\ \ \mu)'V^{-1}\{(\mathbf{1}\ \ \mu)B(\mathbf{1}\ \ \mu)' - V + V\operatorname{tr}[V^{-1}(\mathbf{1}\ \ \mu)B(\mathbf{1}\ \ \mu)' - I_n]\}V^{-1}(\mathbf{1}\ \ \mu)B \\ &= -\frac{n-2}{T-1}B. \end{aligned} \tag{EC32}$$

For Equation (EC26), apply Lemma 1 to (EC12) to find:

$$\begin{aligned} E[S_2(\widehat{B},\delta)] &= \operatorname{tr}[V^{-1}(\mathbf{1}\ \ \mu)B(\mathbf{1}\ \ \mu)' - I_n]B\begin{pmatrix}0\\1\end{pmatrix}(0\ \ 1)B/T + B_{22}B(\mathbf{1}\ \ \mu)'V^{-1}(\mathbf{1}\ \ \mu)B/T \\ &\quad + B\begin{pmatrix}0\\1\end{pmatrix}(0\ \ 1)B(\mathbf{1}\ \ \mu)'V^{-1}(\mathbf{1}\ \ \mu)B/T + B(\mathbf{1}\ \ \mu)'V^{-1}(\mathbf{1}\ \ \mu)B\begin{pmatrix}0\\1\end{pmatrix}(0\ \ 1)B/T. \end{aligned} \tag{EC33}$$

Using (EC22) and the definition of B, (EC33) simplifies to establish (EC26). For Equation (EC27), use (EC25) and (EC26) with the fact that $\widehat{\sigma}_0^2 = (1\ \ \mu_0)\widehat{B}(1\ \ \mu_0)'$ to find

$$\begin{aligned} E[S_2(\widehat{\sigma}_0^2)] &= (1\ \ \mu_0)E[S_2(\widehat{B},\delta) + S_2(\widehat{B},\varepsilon)]\begin{pmatrix}1\\ \mu_0\end{pmatrix} \\ &= (1\ \ \mu_0)\left[\frac{1}{T}B_{22}B - \frac{n-4}{T}B\begin{pmatrix}0\\1\end{pmatrix}(0\ \ 1)B - \frac{n-2}{T-1}B\right]\begin{pmatrix}1\\ \mu_0\end{pmatrix}, \end{aligned} \tag{EC34}$$

which, recognizing that $(1\ \ \mu_0)B(0\ \ 1)' = \xi = B_{22}(\mu_0-\mu_*)$, simplifies to (EC27). For Equation (EC28), by symmetry and using (EC19), we have

$$E[S_2(\widehat{w}'\varepsilon\widehat{w})] = 2w'E[\varepsilon S_1(\widehat{w},\varepsilon)] = 2w'E\{\varepsilon[V^{-1}(\mathbf{1}\ \ \mu)B(\mathbf{1}\ \ \mu)'V^{-1} - V^{-1}]\varepsilon\}w. \tag{EC35}$$

Using Lemma 1 to evaluate the expectation and (EC22) to evaluate a trace, we simplify to find

$$\begin{aligned} E[S_2(\widehat{w}'\varepsilon\widehat{w})] &= 2w'\{[(\mathbf{1}\ \ \mu)B(\mathbf{1}\ \ \mu)' - V] + V\operatorname{tr}[V^{-1}(\mathbf{1}\ \ \mu)B(\mathbf{1}\ \ \mu)' - I_n]\}w/(T-1) \\ &= 2w'\{[(\mathbf{1}\ \ \mu)B(\mathbf{1}\ \ \mu)' - V] + (2-n)V\}w/(T-1) \\ &= 2\left[(1\ \ \mu_0)B\begin{pmatrix}1\\ \mu_0\end{pmatrix} - (n-1)\sigma_0^2\right]\Big/(T-1) = \frac{2}{T-1}[\sigma_0^2 - (n-1)\sigma_0^2] \\ &= -\frac{2(n-2)}{T-1}\sigma_0^2. \end{aligned} \tag{EC36}$$

For Equation (EC29), first note that $E[\delta' S_1(\widehat{w})] = E[\delta' S_1(\widehat{w},\delta)]$ because $E[\delta' S_1(\widehat{w},\varepsilon)] = 0$ by independence of δ and ε, then substitute using (EC18), evaluate the expectation using Lemma 1, and find the trace using (EC22) and (EC23) to obtain

$$\begin{aligned} E[\delta' S_1(\widehat{w},\delta)] &= E\left\{\delta'\left[\xi V^{-1} - V^{-1}(\mathbf{1}\ \ \mu)B\begin{pmatrix}0\\1\end{pmatrix}w' - \xi V^{-1}(\mathbf{1}\ \ \mu)B(\mathbf{1}\ \ \mu)'V^{-1}\right]\delta\right\} \\ &= \frac{1}{T}\operatorname{tr}\left[\xi I_n - V^{-1}(\mathbf{1}\ \ \mu)B\begin{pmatrix}0\\1\end{pmatrix}w'V - \xi V^{-1}(\mathbf{1}\ \ \mu)B(\mathbf{1}\ \ \mu)'\right] \\ &= \frac{1}{T}[n\xi - \xi - 2\xi], \end{aligned} \tag{EC37}$$

which simplifies to (EC29). To prove (EC30), observe that

$$E[S_2(\widehat{w}'V\widehat{w})] = E[S_2(\widehat{w}'\widehat{V}\widehat{w} - \widehat{w}'\varepsilon\widehat{w})] = E[S_2(\widehat{\sigma}_0^2)] - E[S_2(\widehat{w}'\varepsilon\widehat{w})]. \tag{EC38}$$

Substituting using (EC27) and (EC28), then simplifying, establishes (EC30). To prove (EC31), note using $\mu_0 = \widehat{w}'\hat{\mu}$ that

$$\begin{aligned} S_2(\widehat{w}'\mu\mu'\widehat{w}) &= S_2[\widehat{w}'(\hat{\mu}-\mu)(\hat{\mu}-\mu)'\widehat{w} + 2\mu_0\widehat{w}'\mu - \mu_0^2] = S_2[\widehat{w}'\delta\delta'\widehat{w} + 2\mu_0\widehat{w}'\mu - \mu_0^2] \\ &= w'\delta\delta' w + 2\mu_0 S_2(\widehat{w}'\mu). \end{aligned} \tag{EC39}$$

The expectation may be found using Theorem 1 as follows:

$$\begin{aligned} E[S_2(\widehat{w}'\mu\mu'\widehat{w})] &= w'E(\delta\delta')w + 2\mu_0 E[S_2(\widehat{w}'\mu)] \\ &= \frac{w'Vw}{T} - 2\mu_0\frac{n-3}{T}B_{22}(\mu_0 - \mu_*), \end{aligned} \tag{EC40}$$

completing the proof. □

Proof of Theorem 1. Because R_{T+1} is independent of $\widehat{w}$ (and observing that second-order terms containing both δ and ε will have expectation zero), we may write

$$\begin{aligned} E_\Delta(\widehat{w}'\mu) &= E_\Delta(\widehat{w}'\hat{\mu}) - E_\Delta[\widehat{w}'(\hat{\mu}-\mu)] = \mu_0 - E_\Delta(\delta'\widehat{w}) \\ &= \mu_0 - E[S_2(\delta'\widehat{w})] = \mu_0 - E[\delta' S_1(\widehat{w})], \end{aligned} \tag{EC41}$$

which simplifies using (EC29) from Lemma 7 to establish (7). To prove Equation (9), we expand $E_\Delta(\hat{\mu}_{\text{adjusted}})$ to second order as follows:

$$\begin{aligned} E_\Delta\left\{\mu_0 - \frac{n-3}{T}\widehat{B}_{22}(\mu_0 - \hat{\mu}_*)\right\} &= \mu_0 - \frac{n-3}{T}B_{22}(\mu_0 - \mu_*) + E\left\{S_2\left[\mu_0 - \frac{n-3}{T}\widehat{B}_{22}(\mu_0 - \hat{\mu}_*)\right]\right\} \\ &= E_\Delta(\widehat{w}'R_{T+1}) + O\left(\frac{1}{T^2}\right), \end{aligned} \tag{EC42}$$

where the last equality was obtained by observing that expectations of second-order expansion terms with respect to either δ or ε will be $O(1/T)$ by Lemma 1. □

Proof of Theorem 2. Equation (10) is obtained by applying (EC27) from Lemma 7 to the delta-method expansion $E_\Delta(\widehat{\sigma}_0^2) = \sigma_0^2 + E[S_2(\widehat{\sigma}_0^2)]$. Having used iterated expectation to establish the second equality of (11), we next expand as follows:

$$E_\Delta[\widehat{w}'V\widehat{w} + \widehat{w}'\mu\mu'\widehat{w}] - [E_\Delta(\widehat{w}'\mu)]^2 = \sigma_0^2 + E[S_2(\widehat{w}'V\widehat{w})] + \mu_0^2 + E[S_2(\widehat{w}'\mu\mu'\widehat{w})] - [E_\Delta(\widehat{w}'\mu)]^2 \tag{EC43}$$

and then use (EC30), (EC31), and Theorem 1 to evaluate the resulting delta-method expectations, finally simplifying to complete the proof. To prove (13), note that $E_\Delta(\widehat{\sigma}^2_{\text{adjusted}}) = [1 + (2n-3)/T]E_\Delta(\widehat{\sigma}_0^2) + O(1/T^2)$, then substitute from (10) and simplify. □

Proof of Theorem 3. Begin with the naïve frontier that expresses $\widehat{\sigma}_0$ as a function of μ_0:

$$\widehat{\sigma}_0 = \sqrt{\widehat{\sigma}_*^2 + \widehat{B}_{22}(\mu_0 - \hat{\mu}_*)^2}, \tag{EC44}$$

then substitute for $\mu_0 - \hat{\mu}_*$ by solving Equation (8), and substitute for $\widehat{\sigma}_0$ by solving (12). □

Proof of Theorem 4. When $r_f < \hat{\mu}_*$, the naïve Sharpe ratio $S_{\text{naïve}}$, which maximizes $(\mu_0 - r_f)/\widehat{\sigma}_0$ for the naïve frontier (EC44), may be written as

$$S_{\text{naive}} = \sqrt{\frac{1}{\widehat{B}_{22}} + \frac{(\hat{\mu}_* - r_f)^2}{\widehat{\sigma}_*^2}}. \tag{EC45}$$

Because the adjusted frontier remains geometrically a hyperbola with the equation given by (14) instead of (EC44), it follows that we may make the corresponding substitutions in the naïve Sharpe ratio formula (EC45). These substitutions, $\hat{\sigma}_* \to [1+(n-1.5)/T]\hat{\sigma}_*$ and $\hat{B}_{22} \to \hat{B}_{22}[1+(n-1.5)/T]^2 \cdot [1-(n-3)\hat{B}_{22}/T]^{-2}$, establish the first equality in (15). The second equality follows by solving (EC45) for $(\hat{\mu}_* - r_f)^2/\hat{\sigma}_*^2$, substituting, and simplifying. □

Proof of Theorem 5. From the definition of the Sharpe-ratio maximizing portfolio, the naïve CML is defined as

$$\hat{\mu}_{CML(\text{naive})}(r_f) = x r_f + (1-x)\left(\hat{\mu}_* + \frac{\hat{\sigma}_*^2}{\hat{B}_{22}(\hat{\mu}_* - r_f)}\right), \tag{EC46}$$

$$\hat{\sigma}_{CML(\text{naive})}(r_f) = (1-x)\hat{\sigma}_* \sqrt{1 + \frac{\hat{\sigma}_*^2}{\hat{B}_{22}(\hat{\mu}_* - r_f)^2}}. \tag{EC47}$$

Because the adjusted frontier remains geometrically a hyperbola with the equation given by (14) instead of (EC44), it follows that (as in the proof of Theorem 4) we may make corresponding substitutions in the naïve CML formulae to obtain (16) and (17). □

Proof of Corollary to Theorem 5. This follows by reversing Equation (3) to define the naïve target mean as function of the adjusted target mean and then substituting the risky component of Equation (16) for the adjusted Sharpe-ratio maximizing portfolio performance. □

Proof of Theorem 6. For $\hat{w}$ on the estimated efficient frontier (1), we have $\hat{w}'\hat{V}\mathbf{1} = n(1 \quad \hat{\bar{\mu}})\hat{B}\binom{1}{\mu_0}$ and $\hat{\sigma}_0^2 = \hat{w}\hat{V}\hat{w} = (1 \quad \mu_0)\hat{B}\binom{1}{\mu_0}$, where $\hat{\bar{\mu}} = \sum_{i=1}^n \hat{\mu}_i/n$ is the equally weighted average of the estimated asset means. The diversification measure $\widehat{\text{Div}} = (\hat{w} - (1/n)\mathbf{1})'\hat{V}(\hat{w} - (1/n)\mathbf{1})$ can be expanded as follows:

$$\begin{aligned}\widehat{\text{Div}} &= (1 \quad \mu_0)\hat{B}\begin{pmatrix}1\\ \mu_0\end{pmatrix} - 2(1 \quad \mu_0)\hat{B}\begin{pmatrix}1\\ \hat{\bar{\mu}}\end{pmatrix} + \frac{1}{n^2}\mathbf{1}'\hat{V}\mathbf{1}\\ &= \hat{B}_{22}(\mu_0 - \hat{\bar{\mu}})^2 + \frac{1}{n^2}\mathbf{1}'\hat{V}\mathbf{1} - (1 \quad \hat{\bar{\mu}})\hat{B}\begin{pmatrix}1\\ \hat{\bar{\mu}}\end{pmatrix}.\end{aligned} \tag{EC48}$$

Hence, $\widehat{\text{Div}}$ is closer to zero the closer μ_0 is to $\hat{\bar{\mu}}$, ceteris paribus. The target mean that maximizes the naïve Sharpe ratio can be written as follows:

$$\hat{\mu}_{0,\,S-\max-\text{naive}} = \hat{\mu}_* + \frac{\hat{\sigma}_*^2}{\hat{B}_{22}(\hat{\mu}_* - r_f)}. \tag{EC49}$$

Using Equation (19) for the naïve mean corresponding to the adjusted Sharpe-ratio maximizing portfolio, one can see that the target mean for the naïve Sharpe-ratio maximizing portfolio will always be greater than the naïve mean corresponding to the adjusted Sharpe-ratio maximizing portfolio, provided $n > 3$ and $\hat{\mu}_* > r_f$. Hence, the adjusted Sharpe-ratio maximizing portfolio is more diversified whenever

$$\hat{\bar{\mu}} < \frac{\hat{\mu}_{0,\,S-\max-\text{naive}} + \hat{\mu}_{0,\,S-\max-\text{adj}}}{2}. \tag{EC50}$$

Equation (EC50) leads to the third restriction in the theorem statement and completes the proof. □