Dynamic Lot-sizing in Sequential Online Retail Auctions

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Abstract

Retailers often conduct non-overlapping sequential online auctions as a revenue generation and inventory clearing tool. We build a stochastic dynamic programming model for the seller's lot-size decision problem in these auctions. The model incorporates a *random number* of participating bidders in each auction, allows for *any* bid distribution, and is not restricted to any specific price-determination mechanism. Using stochastic monotonicity/stochastic concavity and supermodularity arguments, we present a complete structural characterization of optimal lot-sizing policies under a second order condition on the single-auction expected revenue function. We show that a monotone staircase with unit jumps policy is optimal and provide a simple inequality to determine the locations of these staircase jumps. Our analytical examples demonstrate that the second order condition is met in common online auction mechanisms. We also present numerical experiments and sensitivity analyses using real online auction data.

Keywords: Auctions/bidding, dynamic programming, e-commerce

1. Introduction

Online auctions of retail goods have become a significant component of modern internet commerce. Several large retailers such as Dell (www.dellauction.com) and Sam's Club (auctions.samsclub.com) increasingly use online auctions as a revenue generation mechanism [Bapna et al. 2008; Pinker et al. 2010]. In combination with scrapping excess inventories to firms like Overstock (www.overstock.com), large retailers also use online auctions as an inventory clearing tool. The auction giant eBay (www.ebay.com) and other similar firms such as Ubid (www.ubid.com) provide auction hosting services to retailers like IBM, Sharp and Fujitsu, and also to individual sellers. Companies like Truition (www.truition.com) and ChannelAdvisor (www.channeladvisor.com) specialize in helping businesses conduct online auctions [Odegaard and Puterman 2006]. Based on empirical data available in Vakrat and Seidmann [2000], Pinker et al. [2010] have noted that most retail auction websites conduct a sequence of multi-unit auctions of identical items. These auctions were also observed to be the operational norm by Bapna et al. [2008], Pinker et al. [2003], and Tripathi et al. [2009]. Pinker et al. [2003] have summarized various research issues in such auctions.

Lot-sizes, that is, the number of units to be auctioned in each auction, are one of the key decision variables in sequential auctions [Pinker et al. 2003, 2010; Segev et al. 2001; Tripathi et al. 2009; Vakrat and Seidmann 2000]. A small lot-size induces bidder competition thus increasing the clearing-price. The total revenue may still be lower than one would hope because the number of units sold is small. Uncertainty in the number of participating bidders (demand) in each auction and that in their bids increases decision complexity. For instance, an auction with too large a lot-size may fail due to insufficient demand. Inventory holding costs and the possibility of scrapping inventory to save and recover some of these costs introduce additional economic tradeoffs.

Two papers have investigated inventory scrapping and/or lot-sizing decisions in sequential online retail auctions [Pinker et al. 2010; Tripathi et al. 2009].

Pinker et al. [2010] studied these problems under the following restrictions: a fixed number of participating bidders in each auction, uniform bid distributions with support $[0, 1]$, and a truth revealing multi-unit Vickrey mechanism. These assumptions enabled them to formulate a deterministic dynamic program, wherein a closed-form lot-sizing policy was derived by equating derivatives of value functions to zero within a backward induction procedure. The optimal lot-size decreased at a constant rate from one auction to the next. This rate increased with inventory holding costs and decreased with the number of bidders per auction. In their model, it was optimal to scrap inventory only one time before beginning the entire sequence of auctions.

Tripathi et al. [2009] also assumed a fixed number of participating bidders in each auction, and employed a multi-unit Dutch mechanism. Using uniform bid distributions, they first optimized the lot-size over a sequence of auctions assuming that the lot-size did not change over time. This led to a simple closed-form lot-size expression that resembled the well-known Economic Order Quantity (EOQ) formula in inventory management [Heyman and Sobel 2003]. They also devised a goal programming method to estimate bid distributions from online bid data.

Segev et al. [2001] focused on predicting auction clearing-prices using an orbit queue Markov chain model, and compared these predictions with data obtained from Onsale (www.onsale.com), a Sillicon Valley start-up company. They proposed a deterministic dynamic programming model for lot-size optimization under the restrictive assumption that all items on sale will be sold owing to a sufficiently large number of participating bidders but did not attempt to solve it.

Odegaard and Puterman [2006] considered an auctioneer with two identical items on hand, and determined an optimal time-point at which the second item should be "released" for an auction. They derived conditions to ensure an optimal control-limit release-time policy. This control-limit was decreasing in holding cost.

Vulcano et al. [2002] studied a problem motivated by airline ticket selling websites like Priceline (www.priceline.com). The seller first observed bids from potential travelers, and then chose how many and which bids to accept, as opposed to publicly pre-committing lot-sizes at the beginning of each auction before receiving bids as practiced in retail auctions [Odegaard and Puterman 2006; Pinker et al. 2010; Segev et al. 2001; Tripathi et al. 2009]. Consequently, they solved a variable supply allocation problem rather than a lot-size optimization problem to obtain a structural result similar to ours but utilized different mathematical analysis and sufficient conditions as developed by Myerson [1981] and Maskin and Riley [2000]. This work was later extended to an infinite-horizon joint auctioning and pricing problem under holding and ordering costs [van Ryzin and Vulcano 2004].

The basic setting in our paper is similar to Pinker et al. [2010] and Tripathi et al. [2009] in that we consider a seller who conducts a sequence of non-overlapping online auctions of retail goods. However, in contrast to their work, we incorporate uncertainty in the number of participating bidders (stochastic demand) in each auction; do not restrict our formulation to any specific clearing-price determination rule; and allow for any bid distribution (see Section 2 for details). To the best of our knowledge, this is the first paper that successfully overcomes all mathematical difficulties introduced by this generalization in the retail pre-committing setting to provide a complete structural characterization of optimal inventory scrapping and lot-sizing policies as in Theorem 2.1.

More specifically, under the second order condition (7) on the single-auction expected revenue function, we show that a threshold inventory-scrapping policy, and a monotone staircase with unit jumps lot-sizing policy are optimal. This condition roughly requires that the marginal single-auction expected revenue, normalized by the probability of sufficient number of bidders participating, be decreasing in lot-size. It is then optimal to scrap all inventory above a time-dependent threshold inventory level, and not to scrap any inventory below the threshold. This threshold equals the inventory level at which the scrap-value of a unit exceeds its marginal value over all remaining auctions. Moreover, if lot-size x is optimal in post-scrapping inventory i, then either lot-size x or lot-size $x+1$ is optimal in post-scrapping inventory $i+1$. This unit jump in optimal lot-size occurs when the normalized marginal single-auction expected revenue exceeds the discounted marginal value of saving the additional unit for future auctions. See Theorem 2.1 and its proof in Section 3 for precise detailed versions of these statements. Section 3.1 includes several examples where our second order condition is met. Limitations and potential extensions of our model are discussed in Section 5.

2. Problem description and mathematical formulation

Consider a seller with some initial inventory of identical units on hand. We assume that the seller conducts a sequence of $1 \leq T < \infty$ auctions indexed by $t = 1, 2, \ldots, T$. The seller uses a fixed auction mechanism in all auctions and this mechanism is disclosed to the bidders. Examples of auction mechanisms include multi-unit Vickrey as on eBay, multi-unit Dutch as on Sam's Club, and Yankee as on Ubid.

Under stochastic demand, one-shot scrapping as in Pinker et al. [2010] may not be optimal; in fact, it may lead to negative marginal values. It is essential to dynamically exploit the flexibility to scrap inventory *even if* the scrap-value is zero. Thus, at the beginning of auction t, the seller makes two decisions after observing inventory i on hand: (i) the number of units y to be scrapped for a value of $s \geq 0$ per unit, and (ii) of the $i - y$ remaining units, the lot-size x to be put up for the tth auction. This lot-size is disclosed to the potential bidders at the start of the tth auction.

A random number N of bidders who wish to buy one unit each then place their bids. The probability mass function (pmf) of N is denoted $g(\cdot)$, its support is denoted $\mathcal{N} \subseteq \mathbb{N}_+$, and its distribution function is denoted $G(\cdot)$. Consistent with the existing literature, we assume that bidders are independent across auctions, and identical both within an auction and across auctions. A detailed discussion of practical limitations introduced by this assumption is included, for instance, in Section 4 of Pinker et al. [2010], and we do not repeat it here (also see Section 5). More specifically, the final bids in all auctions are independent and identical (iid) random variables B with distribution $F(\cdot)$, finite expectation, and support $\mathcal{B} \subseteq \mathbb{R}_+$ whose smallest element is denoted L. The existence of a probability density function for B is not assumed, and in particular, B may be discrete. We remark that our setting is flexible and general enough to allow bid distributions that are statistically estimated using online data, those derived from game theoretic analyses of how bidders might behave in sequential online multi-unit auctions, and the ones obtained through a combination of these two approaches (see Bapna et al. [2002, 2003, 2008]; Fatima [2008]; Jiang and Leyton-Brown [2007]; Pinker et al. [2010]; Tripathi et al. [2009] for examples of such techniques).

If the actual number of bidders n in an auction is more than the lot-size x, the seller generates revenue through bidder competition by selling all x units. We denote this revenue by $\pi(x;n)$, and it is derived from the expected value of a mechanism-specific order-statistic of $F(\cdot)$. See the beginning of Section 3.1, and in particular, Equations (17)-(19), for specific examples of $\pi(x;n)$. If $n \leq x$, the auction fails due to a lack of bidder competition. Note that this scenario does not arise in Pinker et al. [2010] and Tripathi et al. [2009] owing to their assumption of deterministic demand. When an auction fails, the seller sells one unit to each of the n bidders for amount L [Pinker et al. 2003]. Equivalently, in the language of Pinker et al. [2003], the "minimum bid" of the auction is set to L. This is better than selling for any price less than L. The seller may however benefit from using a minimum bid requirement of some $\Lambda > L$, and then selling each unit in a failed auction for Λ. This introduces challenging tradeoffs, which require dynamic optimization of Λ and are not the focus of this paper (also see Section 5). Another entirely different possibility for the seller is to cancel a failed auction, returning the x units originally intended for sale back to the inventory held. Canceling auctions disappoints bidders who did participate, leading to negative feedback from these unsatisfied bidders. This hurts the seller's reputation that is critical for success in e-commerce [Ba and Pavlou 2002; Resnick et al. 2006]. We therefore do not follow this alternative approach.

The holding cost of carrying each unit in inventory during auction t is denoted $h \geq 0$, and it is assumed to be incurred for the $i-y$ units remaining after scrapping y units from inventory i at the beginning of auction t. As in classic dynamic lot-sizing models [Denardo 2003], we assume that inventory remaining after the Tth auction has no value; our results generalize in a straightforward manner to the case where terminal inventory fetches concave value.

The seller's goal is to maximize total discounted expected profit obtained through the T auctions where the discount factor equals $0 < \alpha < 1$. A dynamic programming model for the seller's problem is developed in the next section.

2.1. Dynamic programming model

First, a note on terminology. Since most quantities of interest such as inventory i and lot-size x in this paper are integer valued, we use $i \geq 0$ to imply $i = 0, 1, 2, \ldots$. The symbol $\triangleq$ is reserved for defining some of the mathematical notation. We recall relevant terminology for functions $\varphi(\cdot)$ of non-negative integers. Function $\varphi(\cdot)$ is said to be increasing at x, if $\varphi(x+1) \geq \varphi(x)$, that is, when the first difference $\Delta\varphi(x) \triangleq \varphi(x+1) - \varphi(x) \geq 0$. Similarly for decreasing. We also define the second difference $\Delta^2\varphi(x) \triangleq \Delta(\Delta\varphi(x)) = [\varphi(x+2) - \varphi(x+1)] - [\varphi(x+1) - \varphi(x)]$. We say that $\varphi(\cdot)$ is concave[1] over the set $J = \{0, 1, 2, \ldots, j\}$

[1]See condition S2-15 in Denardo [2003], and also Lemma 6 therein for an equivalence between concavity of functions over non-negative integers and that of their extensions to $\mathbb{R}_+$ using linear interpolation.

for some integer $j \geq 2$ if $\Delta^2\varphi(x) \leq 0$ for each $x \in \{0, 1, \ldots, j-2\}$. We say that a function is concave if its second difference is non-positive at every non-negative integer. Similarly for increasing and decreasing.

Let $\phi(\cdot)$ denote the expected revenue from one auction as a function of the lot-size. Based on the above problem description, we have,

$$\phi(x) \triangleq \underbrace{\sum_{n=0}^{x} nLg(n)}_{\text{failed auction}} + \underbrace{\sum_{n=x+1}^{\infty} g(n)\pi(x;n)}_{\text{successful auction}}, \text{ for } x \geq 0^2. \tag{1}$$

It often helps to imagine the extension

$$\bar{\pi}(x;n) \triangleq \begin{cases} \pi(x;n) & \text{for } n > x \\ nL & \text{for } n \leq x, \end{cases} \tag{2}$$

which leads to the compact formula $\phi(x) = E[\bar{\pi}(x;N)] = \sum_{n=0}^{\infty} g(n)\bar{\pi}(x;n)$.

The following structural observations about $\phi(\cdot)$ hint at the nature of mathematical difficulties introduced by stochastic demand. Even when $\pi(\cdot;n)$ is concave in lot-size for a fixed value of n, the function $\bar{\pi}(\cdot;n)$ is not concave because it is flat (independent of x) for $x \geq n$. A possible overall effect is that when the expectation over the pmf of the number of participating bidders is taken, the standard approach of "expectation of a concave function is concave" cannot be used to establish concavity of $\phi(\cdot)$ in lot-size (recall that concavity of revenue functions often facilitates analytical results [Talluri and van Ryzin 2005]). Indeed, in a multi-unit Dutch auction with a discrete-uniform number of participating bidders and uniformly distributed bids, the expected revenue is not concave in lot-size. Moreover, in our experience with several examples, it is impossible to derive a closed-form formula for a maximizer of the single-auction expected revenue function owing to algebraic difficulties *even if* we were to make the approximating assumption that non-integer lot-sizes are feasible. As a result, even in a single auction, deciding an optimal lot-size is non-trivial and must be done numerically. This is in contrast with Pinker et al. [2010] and Tripathi et al. [2009], where closed-form optimal lot-sizes are readily obtained by equating first derivatives to zero under their approximating assumption that non-integer lot-sizes were feasible. Fortunately, we are able to overcome all such hurdles to derive the structure of optimal policies.

We define the random variable $\zeta_x = \min\{x, N\}$. It represents the drop in inventory after one auction — x when the auction is successful, that is, when $N > x$; and N when the auction fails, that is, when $N \leq x$. For $i \geq 0$, and $1 \leq t \leq T$, let $V_t(i)$ denote the maximum total expected profit generated in auctions t through T when the inventory on hand beginning auction t is i. That is, $V_t(\cdot)$ are the optimal value functions. Bellman's equations for the seller's problem are given by

$$V_t(i) = \max_{\substack{0 \leq y \leq i \\ 0 \leq x \leq i-y}} \Big[sy - h(i-y) + \phi(x) + \alpha E\big[V_{t+1}(i-y-\zeta_x)\big]\Big], \text{ for } i \geq 0,\ 1 \leq t \leq T, \tag{3}$$

[2]When $\mathcal{N}$ is finite, the domain of $\phi(\cdot)$ can and will be restricted without loss of optimality to $0 \leq x \leq M$, where M is the largest element of $\mathcal{N}$.

with the boundary condition $V_{T+1}(i) = 0$ for $i \geq 0$. Note also that $V_t(0) = 0$ for all $1 \leq t \leq T$. It is convenient to work with (3) rewritten in the following nested form:

$$V_t(i) = \max_{0 \leq y \leq i} \Big[sy + U_t(i-y) \Big], \ i \geq 1, \ 1 \leq t \leq T, \text{ where,} \tag{4}$$

$$U_t(j) = \max_{0 \leq x \leq j} \Big[-hj + \phi(x) + \alpha E\big[V_{t+1}(j - \zeta_x)\big] \Big], \ j \geq 1, \ 1 \leq t \leq T. \tag{5}$$

Note that $U_t(0) = 0$ for $1 \leq t \leq T$.

For lot-size x, let

$$\partial\Phi(x) \triangleq \frac{\Delta\phi(x)}{\sum\limits_{n=x+1}^{\infty} g(n)} = \frac{\Delta\phi(x)}{1 - G(x)}. \tag{6}$$

This is the marginal single-auction expected revenue normalized by the probability that the auction is successful owing to a sufficient number of bidders participating. Also let $\partial^2\Phi(x) \triangleq \partial\Phi(x+1) - \partial\Phi(x)$. In the next section, we prove the following key result:

Theorem 2.1. *Let $\hat{x}$ denote the smallest maximizer of $\phi(\cdot)$. Suppose*

$$\partial^2\Phi(x) \leq 0, \ \textit{for } x \in \{0, 1, \ldots, \hat{x} - 2\}. \tag{7}$$

Then

1. *A threshold policy is optimal for scrapping inventory — suppose $0 \leq i_t^*$ is the smallest inventory i for which $s > U_t(i+1) - U_t(i)$; then in any inventory $i \geq 1$, it is optimal to scrap $(i - i_t^*)^+ \triangleq \max\{0, i - i_t^*\}$ units. That is, we do not scrap any units when inventory on hand is i_t^* or less; we scrap all units over and above i_t^* when inventory on hand is more than i_t^*.*
2. *A monotone "staircase with unit jumps" policy is optimal for lot-sizing — suppose x is the smallest lot-size optimal in inventory j; then either x or $x+1$ is the smallest lot-size optimal in inventory $j+1$. In particular, the smallest optimal lot-size jumps from x to $x+1$ at the smallest inventory level $k \geq j+1$ for which*

$$\partial\Phi(x) > \alpha(V_{t+1}(k-x) - V_{t+1}(k-x-1)). \tag{8}$$

The discussion below provides insight into inequality (7).

Lemma 2.2. *Condition (7) implies that $\phi(\cdot)$ is concave over $\{0, 1, 2, \ldots, \hat{x}\}$.*

Proof. Since $\hat{x}$ is the smallest maximizer of $\phi(\cdot)$, $\phi(\hat{x}) > \phi(\hat{x}-1)$. In addition, because both $\sum\limits_{n=x+1}^{\infty} g(n)$ and $\sum\limits_{n=x+2}^{\infty} g(n)$ are positive, condition (7) implies that $\Delta\phi(x) \geq 0$ for $0 \leq x \leq \hat{x}-1$. Since $\sum\limits_{n=x+1}^{\infty} g(n) \geq \sum\limits_{n=x+2}^{\infty} g(n)$, this implies that $\phi(\cdot)$ is concave over $\{0, 1, 2, \ldots, \hat{x}\}$. □

Remark 2.3. *Inequality (7) does not imply concavity at x when $\Delta\phi(x)$ and $\Delta\phi(x+1)$ are negative; indeed, for some of our examples in Section 3.1, condition (7) is met over the entire domain of $\phi(\cdot)$, and yet, $\phi(\cdot)$ is not concave over this domain but is concave over $\{0, 1, 2, \ldots, \hat{x}\}$.*

Lemma 2.4. *If inequality (7) holds over the entire domain of $\phi(\cdot)$, then $\phi(\cdot)$ is unimodal, that is, increasing up to $\hat{x}-1$ and then decreasing.*

Proof. Because $\hat{x}$ is a maximizer of $\phi(\cdot)$, $\phi(\hat{x}) \geq \phi(\hat{x}+1)$, that is, $0 \geq \Delta\phi(\hat{x})$. Inequality (7) then implies that $0 \geq \Delta\phi(x)$ for all $x \geq \hat{x}$. Moreover, as shown in Lemma 2.2, $\Delta\phi(x) \geq 0$ for $0 \leq x \leq \hat{x}-1$. Hence $\phi(\cdot)$ is unimodal. □

3. Structural analysis of optimal policies

We first prove a few preliminary results that are later used in establishing Theorem 2.1. The following Lemma shows that the seller can exploit the opportunity to scrap inventory at the beginning of each auction to ensure that the marginal value of each unit is non-negative.

Lemma 3.1. *For every auction t, $V_t(i+1) - V_t(i) \geq s$, and hence $V_t(\cdot)$ is increasing in inventory.*

Proof. Consider any auction $1 \leq t \leq T$, and any inventory $i \geq 0$. Suppose it is optimal to scrap y units if the inventory on hand beginning auction t equals i. Then it is feasible to scrap $y+1$ units when the inventory on hand beginning auction t is $i+1$. Therefore,

$$V_t(i+1) \geq s(y+1) + U_t(i+1-(y+1)) = sy + s + U_t(i-y) = s + V_t(i).$$

That is, $V_t(i+1) - V_t(i) \geq s$ as required. □

Stochastic monotonicity and stochastic concavity properties [Shaked and Shanthikumar 1994] of the random variable ζ_x are used in our structural analysis. We now precisely define and establish these. Let $G_x(\cdot)$ denote the distribution function of ζ_x. That is,

$$G_x(w) \triangleq P(\zeta_x \leq w) = \begin{cases} 0 \text{ for } w < 0 \\ P(N \leq \lfloor w \rfloor) \text{ for } 0 \leq w < x \\ 1 \text{ for } w \geq x. \end{cases} \tag{9}$$

Also, $\bar{G}_x(w) \triangleq 1 - G_x(w)$ defines $P(\zeta_x > w)$. Recall that ζ_x is said to be stochastically increasing (in x) if for any x_1 and x_2 with $x_1 \geq x_2$, $\bar{G}_{x_1}(w) \geq \bar{G}_{x_2}(w)$ for every fixed real number w (Definition B.1 page 640 of Talluri and van Ryzin [2005]). Also, ζ_x is stochastically increasing in x if and only if for any real valued, increasing function $f(\cdot)$, $E[f(\zeta_x)]$ is increasing in x (Proposition B.2 page 640 of Talluri and van Ryzin [2005]). Moreover, ζ_x is said to be stochastically concave if for any real valued, concave function $f(\cdot)$, $E[f(\zeta_x)]$ is concave in x (Definition B.2 page 640 of Talluri and van Ryzin [2005]). In addition, ζ_x is said to be strongly stochastically concave if $\zeta_x = \Omega(x, Z)$ where Z is a random variable independent of x and Ω is concave in x for every value of Z (Definition B.3 page 640 of Talluri and van Ryzin [2005]). A strongly stochastically concave random variable is stochastically concave (Proposition B.3 page 641 of Talluri and van Ryzin [2005]).

Lemma 3.2. *The random variable ζ_x is stochastically increasing and stochastically concave in x.*

Proof. Provided in Appendix Appendix A.1. □

For brevity, let $Q_t(j,x) \triangleq -hj + \phi(x) + \alpha E\left[V_{t+1}(j-\zeta_x)\right]$, for $j \geq 0$, $0 \leq x \leq j$, $1 \leq t \leq T$. Recall that $\hat{x}$ denotes the smallest maximizer of the single auction expected revenue function $\phi(\cdot)$. Lemma 3.3 below shows that we can restrict attention to lot-sizes less than or equal to $\hat{x}$ *without loss of optimality in problem (5).* This conclusion is important because it allows us to focus on lot-sizes over which $\phi(\cdot)$ is concave under inequality (7). This proves critical in establishing that Bellman's dynamic programming operation in (3) preserves optimal value function concavity — a crucial intermediate step in proving Theorem 2.1. Our proof of Lemma 3.3 uses monotonicity of optimal value functions established in Lemma 3.1 and stochastic monotonicity of ζ_x from Lemma 3.2.

Lemma 3.3. *For every auction t, and any inventory $j \geq 1$, there exists a lot-size $x_t^*(j)$ optimal in inventory j in problem (5) such that $x_t^*(j) \leq \hat{x}$. In particular, the smallest lot-size optimal in $j \geq 1$ is not more than $\hat{x}$.*

Proof. Suppose lot-size $\hat{x}+k$ for some integer $k \geq 1$ is feasible in inventory j. We have

$$Q_t(j,m) - Q_t(j,\hat{x}+k) = \phi(\hat{x}) - \phi(\hat{x}+k) + \alpha E\big[V_{t+1}(j-\zeta_{\hat{x}})\big] - \alpha E\big[V_{t+1}(j-\zeta_{\hat{x}+k})\big] \geq 0$$

because (i) $\phi(\hat{x}) \geq \phi(\hat{x}+k)$ as $\hat{x}$ is a maximizer of $\phi(\cdot)$, and (ii) $E\big[V_{t+1}(j-\zeta_{\hat{x}})\big] \geq E\big[V_{t+1}(j-\zeta_{\hat{x}+k})\big]$ as $V_{t+1}(\cdot)$ is increasing by Lemma 3.1 and ζ_x is stochastically increasing by Lemma 3.2. Thus, auctioning more than $\hat{x}$ does not fetch any additional benefit as compared to auctioning $\hat{x}$. Consequently, there exists a lot-size $x_t^*(j) \leq \hat{x}$ that solves problem (5) for inventory j. □

Since $V_{T+1}(i) = 0$ for all $i \geq 0$, $V_{T+1}(\cdot)$ is trivially concave. Theorem 2.1 then follows by induction from the sequence of results established below.

The proof of the first conclusion in Proposition 3.4 below uses stochastic monotonicity of ζ_x from Lemma 3.2, whereas the second conclusion relies on stochastic concavity of ζ_x from Lemma 3.2 and concavity of $\phi(\cdot)$ over $\{0,1,2,\ldots,\hat{x}\}$ implied by condition (7).

Proposition 3.4. *Suppose that $V_{t+1}(\cdot)$ is concave. Suppose lot-size x_1 is feasible in inventory j in problem (5) for auction t, that is, $x_1 \leq j$, and let lot-size x_2 be such that $x_2 < x_1$. Then*

$$Q_t(j+1,x_2) - Q_t(j,x_2) \leq Q_t(j+1,x_1) - Q_t(j,x_1), \tag{10}$$

that is, $Q_t(\cdot,\cdot)$ has increasing differences. Moreover, for every inventory $j \geq 0$,

$$Q_t(j,x+1) + Q_t(j,x+1) \geq Q_t(j,x+2) + Q_t(j,x), \tag{11}$$

that is, $Q_t(j,\cdot)$ is concave over $\{0,1,2,\ldots,\min\{j,\hat{x}\}\}$.

Proof. Consider the function $f(z) \triangleq V_{t+1}(j+1-z) - V_{t+1}(j-z)$. This function is increasing over $z \in \{0,1,\ldots,j-1\}$ because $V_{t+1}(\cdot)$ is concave. Hence, because ζ_x is stochastically increasing by Lemma 3.2, $E[f(\zeta_{x_2})] \leq E[f(\zeta_{x_1})]$ as $x_2 < x_1$. Therefore,

$$\begin{aligned}
&Q_t(j+1,x_2) - Q_t(j,x_2)\\
&\qquad= -hj - h + \phi(x_2) + \alpha E[V_{t+1}(j+1-\zeta_{x_2})] + hj - \phi(x_2) - \alpha E[V_{t+1}(j-\zeta_{x_2})]\\
&\qquad= -h + \alpha E[f(\zeta_{x_2})] \leq -h + \alpha E[f(\zeta_{x_1})]\\
&\qquad= -h + \alpha E[V_{t+1}(j+1-\zeta_{x_1})] - \alpha E[V_{t+1}(j-\zeta_{x_1})]\\
&\qquad= -hj - h + \phi(x_1) + \alpha E[V_{t+1}(j+1-\zeta_{x_1})] + hj - \phi(x_1) - \alpha E[V_{t+1}(j-\zeta_{x_1})]\\
&\qquad= Q_t(j+1,x_1) - Q_t(j,x_1).
\end{aligned}$$

This establishes (10).

Fix any inventory $j \geq 0$. Consider the function $V_{t+1}(j-z)$ and note that it is concave over $z \in \{0, 1, \ldots, j\}$. Thus $E[V_{t+1}(j-\zeta_x)]$ is concave over $x \in \{0, 1, \ldots, j\}$ because ζ_x is stochastically concave by Lemma 3.2. Thus, because $\phi(\cdot)$ is concave over $\{0, 1, 2, \ldots, \hat{x}\}$, $Q_t(j, x) = -hj + \phi(x) + \alpha E[V_{t+1}(j-\zeta_x)]$ is also concave in x over $\{0, 1, 2, \ldots, \min\{j, \hat{x}\}\}$. □

The next corollary establishes monotonicity of optimal lot-sizes in inventory on hand using the increasing differences property of $Q_t(\cdot, \cdot)$ from Proposition 3.4.

Corollary 3.5. *Suppose that $V_{t+1}(\cdot)$ is concave. Suppose x_1 is the smallest lot-size optimal in inventory j, and x_2 is the smallest lot-size optimal in inventory $j+1$ in problem (5) for auction t. Then $x_2 \geq x_1$.*

Proof. Suppose not, that is, $x_2 < x_1$. Then $Q_t(j, x_1) > Q_t(j, x_2)$ (the inequality is strict because x_1 is the smallest lot-size optimal in j), and $Q_t(j+1, x_2) \geq Q_t(j+1, x_1)$ (the inequality is not strict because both x_1 and x_2 could be optimal in $j+1$). Adding the two inequalities and rearranging terms, we get $Q_t(j+1, x_2) - Q_t(j, x_2) > Q_t(j+1, x_1) - Q_t(j, x_1)$, which contradicts inequality (10) established in Proposition 3.4. □

Proposition 3.6 below further strengthens monotonicity of optimal lot-sizes from Corollary 3.5 — when the optimal lot-size increases, it increases by one. The proof uses Corollary 3.5, concavity of $Q_t(j, \cdot)$ established in Proposition 3.4, and condition (7). In addition to being an interesting structural result in itself, this Proposition later plays a critical role in the proof of Proposition 3.9 where we show that optimal value function concavity is preserved by Bellman's dynamic programming operator in problem (5).

Proposition 3.6. *Suppose that $V_{t+1}(\cdot)$ is concave and x is the smallest lot-size optimal in inventory j in problem (5) for auction t. Then either x or $x+1$ is the smallest lot-size optimal in inventory $j+1$ in problem (5) for auction t.*

Proof. If $x = j$, then by Corollary 3.5, either x or $x+1$ must be the smallest lot-size optimal in $j+1$. So we focus on the more challenging case where $x < j$ and thus $x+1$ is feasible in j and $x+2$ is feasible in $j+1$. We first show that $Q_t(j+1, x+1) \geq Q_t(j+1, x+k)$ for all $k \geq 2$ such that $x + k \leq \min\{\hat{x}, j+1\}$ (recall from Lemma 3.3 that in our search for the smallest lot-size optimal in $j+1$, we do not need to look beyond $\hat{x}$). Toward that end, we first demonstrate that $Q_t(j+1, x+1) \geq Q_t(j+1, x+2)$. For if not, then

$$\partial\Phi(x+1) > \alpha\left[V_{t+1}(j-x) - V_{t+1}(j-x-1)\right]. \tag{12}$$

But note that $Q_t(j, x) \geq Q_t(j, x+1)$ as x is optimal in j and $x+1$ is feasible in j and therefore

$$\alpha\left[V_{t+1}(j-x) - V_{t+1}(j-x-1)\right] \geq \partial\Phi(x). \tag{13}$$

Inequalities (12) and (13) imply that $\partial\Phi(x+1) > \partial\Phi(x)$, contradicting (7). Moreover, $Q_t(j+1, x+1) \geq Q_t(j+1, x+2)$ and concavity of $Q_t(j+1, \cdot)$ in its second argument as in (11) from Proposition 3.4 imply that $Q_t(j+1, x+1) \geq Q_t(j+1, x+k)$ for all $k \geq 2$ such that $x + k \leq \min\{m, j+1\}$ as required. The above discussion, in view of Corollary 3.5, implies that either x or $x+1$ is the smallest lot-size optimal in $j+1$. □

The above proposition affirms that when the optimal lot-size increases, it increases by one; it does not however provide the location, that is, the inventory level at which this unit jump in the optimal lot-size occurs. Inequality (14) in Corollary 3.7 below ties this loose end.

Corollary 3.7. *Suppose that $V_{t+1}(\cdot)$ is concave and x is the smallest lot-size optimal in inventory j in problem (5) for auction t. Then the smallest optimal lot-size jumps to $x+1$ at the smallest inventory level $k \geq j+1$ for which*

$$\partial\Phi(x) > \alpha(V_{t+1}(k-x) - V_{t+1}(k-x-1)). \tag{14}$$

Proof. Proposition 3.6 implies that x remains the smallest lot-size optimal in $j+1$ if and only if $Q_t(j+1, x) \geq Q_t(j+1, x+1)$, that is, if and only if $\partial\Phi(x) \leq \alpha(V_{t+1}(j+1-x) - V_{t+1}(j-x))$. If x does remain the smallest lot-size optimal in $j+1$, then it is the smallest lot-size optimal in $j+2$ if and only if $\partial\Phi(x) \leq \alpha(V_{t+1}(j+2-x) - V_{t+1}(j+1-x))$. Continuing this argument, we see that the smallest optimal lot-size will jump to $x+1$ at the smallest inventory level $k \geq j+1$ for which the strict inequality (14) is satisfied (the left hand side in (14) does not depend on the inventory, whereas, by concavity of $V_{t+1}(\cdot)$, the right hand side decreases as k increases). ☐

The next important step in the proof is to show that concavity of the optimal value function $V_{t+1}(\cdot)$ is carried over to $U_t(\cdot)$ by Bellman's dynamic programming operation in problem (5). A simple algebraic proof does not appear possible. We therefore define a new random variable $\gamma_j = \zeta_{x+j} - j = \min\{x+j, N\} - j$ for a fixed x (our notation γ_j suppresses dependence of γ_j on x for brevity), and exploit its stochastic monotonicity and stochastic concavity properties that we first establish in Lemma 3.8.

Lemma 3.8. *The random variable γ_j is stochastically decreasing and stochastically concave in j.*

Proof. Provided in Appendix Appendix A.2. ☐

The proof of the proposition below uses Lemma 3.3, Corollary 3.5, Proposition 3.6, and concavity of $\phi(\cdot)$ over $\{0, 1, 2, \ldots, \hat{x}\}$ implied by condition (7).

Proposition 3.9. *Suppose that $V_{t+1}(\cdot)$ is concave. Then $U_t(\cdot)$ is concave.*

Proof. We want to show that

$$U_t(j+1) + U_t(j+1) \geq U_t(j+2) + U_t(j).$$

Suppose x is the smallest lot-size optimal in inventory j and y is the smallest lot-size optimal in inventory $j+2$ in problem (5) for auction t. Then by Lemma 3.3, $y \leq \hat{x}$. By Corollary 3.5, $x \leq y$. By Proposition 3.6, $y \leq x+2$. We consider three cases:

Case 1: $j + 2 > y = x$.

$$
\begin{aligned}
&U_t(j+1) + U_t(j+1)\\
&= Q_t(j+1, x) + Q_t(j+1, x)\\
&= -hj - h + \phi(x) + \alpha E[V_{t+1}(j+1-\zeta_x)] - hj - h + \phi(x) + \alpha E[V_{t+1}(j+1-\zeta_x)]\\
&\geq -hj - 2h + \phi(x) + \alpha E[V_{t+1}(j+2-\zeta_x)] - hj + \phi(x) + \alpha E[V_{t+1}(j-\zeta_x)]\\
&= -hj - 2h + \phi(y) + \alpha E[V_{t+1}(j+2-\zeta_y)] - hj + \phi(y) + \alpha E[V_{t+1}(j-\zeta_y)]\\
&= Q_t(j+2, y) + Q_t(j, y) = U_t(j+2) + U_t(j),
\end{aligned}
$$

where the inequality follows from concavity of $V_{t+1}(\cdot)$.
Case 2: $j + 2 > y = x + 1$.
First note that $V_{t+1}(j+1-z) - V_{t+1}(j-z)$ is increasing in z because $V_{t+1}(\cdot)$ is concave. Therefore, $E[V_{t+1}(j+1-\gamma_k)] - E[V_{t+1}(j-\gamma_k)]$ is decreasing in k as γ_k is stochastically decreasing. In particular, $E[V_{t+1}(j+1-(\gamma_1))] - E[V_{t+1}(j-(\gamma_1))]$ is bounded above by $E[V_{t+1}(j+1-(\gamma_0))] - E[V_{t+1}(j-(\gamma_0))]$. That is,

$$
E[V_{t+1}(j+2-\zeta_{x+1})] - E[V_{t+1}(j+1-\zeta_{x+1})] \leq E[V_{t+1}(j+1-\zeta_x)] - E[V_{t+1}(j-\zeta_x)]. \quad (15)
$$

Then we have

$$
\begin{aligned}
&U_t(j+1) + U_t(j+1)\\
&\geq Q_t(j+1, x+1) + Q_t(j+1, x)\\
&= -hj - h + \phi(x+1) + \alpha E[V_{t+1}(j+1-\zeta_{x+1})] - hj - h + \phi(x) + \alpha E[V_{t+1}(j+1-\zeta_x)].
\end{aligned}
$$

Inequality (15) implies that this expression is bounded below by

$$
\begin{aligned}
&\geq -hj + \phi(x) + \alpha E[V_{t+1}(j-\zeta_x)] - hj - 2h + \phi(x+1) + \alpha E[V_{t+1}(j+2-\zeta_{x+1})]\\
&= Q_t(j, x) + Q_t(j+2, x+1) = U_t(j) + U_t(j+2).
\end{aligned}
$$

Case 3: $j + 2 \geq y = x + 2$.
This implies that $x + 1$ is optimal in $j + 1$. The function $V_{t+1}(j - z)$ is concave in z. This implies that $E[V_{t+1}(j-\gamma_k)]$ is concave in k because γ_k is stochastically concave in k. In particular,

$$
E[V_{t+1}(j-(\zeta_{x+1}-1))] + E[V_{t+1}(j-(\zeta_{x+1}-1))] \geq E[V_{t+1}(j-(\zeta_{x+2}-2))] + E[V_{t+1}(j-\zeta_x)]. \quad (16)
$$

Therefore, $U_t(j+1) + U_t(j+1)$ can be bounded below as follows.

$$
\begin{aligned}
&U_t(j+1) + U_t(j+1) = Q_t(j+1, x+1) + Q_t(j+1, x+1)\\
&= -hj - h + \phi(x+1) + \alpha E[V_{t+1}(j+1-\zeta_{x+1})] - hj - h + \phi(x+1) + \alpha E[V_{t+1}(j+1-\zeta_{x+1})]\\
&\geq -hj + \phi(x) + \alpha E[V_{t+1}(j+1-\zeta_{x+1})] - hj - 2h + \phi(x+2) + \alpha E[V_{t+1}(j+1-\zeta_{x+1})]\\
&\geq -hj + \phi(x) + \alpha E[V_{t+1}(j-\zeta_x)] - hj - 2h + \phi(x+2) + \alpha E[V_{t+1}(j+2-\zeta_{x+2})]\\
&= Q_t(j, x) + Q_t(j+2, x+2) = U_t(j) + U_t(j+2),
\end{aligned}
$$

where the first inequality follows from concavity of $\phi(\cdot)$ over $\{0, 1, 2, \ldots, \hat{x}\}$, and the second inequality from (16). This completes the proof. □

The focus thus far was on problem (5). Now we consider problem (4). We define the function $\Psi_t(\cdot,\cdot)$ for convenience as follows:

$$\Psi_t(i,y) \triangleq sy + U_t(i-y), \text{ for } i \geq 0,\ 0 \leq y \leq i, \text{ and } 1 \leq t \leq T.$$

The next Lemma is an intermediate step in proving Proposition 3.11.

Lemma 3.10. *The optimal inventory scrapping policy in problem (4) for auction t has the following structural properties:*

1. *Suppose it is optimal to scrap y units in inventory i. Then in inventory $i+1$, it is at least as profitable to scrap $y+1$ units as it is to scrap $1 \leq z \leq i+1$ units.*
2. *Suppose $U_t(\cdot)$ is concave for auction t. Suppose that in some inventory $i \geq 1$, scrapping $y > 0$ units is at least as profitable as not scrapping any units. Then the same is true for all inventories $i' \geq i$.*

Proof. Provided in Appendix Appendix A.3. □

Proposition 3.11. *Suppose that $U_t(\cdot)$ is concave for auction t. Suppose $i_t^* \geq 0$ is the smallest inventory i for which $s > U_t(i+1) - U_t(i)$. Then in any inventory $i \geq 1$, it is optimal to scrap $(i - i_t^*)^+ \triangleq \max\{0, i - i_t^*\}$ units. Moreover, $V_t(\cdot)$ is concave for auction t.*

Proof. Because of the first claim in Lemma 3.10, $y = 0$ is optimal in all inventories $i' \leq i_t^*$, and $y = 1$ is optimal in inventory $i_t^* + 1$. Repeated application of the two claims in Lemma 3.10 then ensures that $y = 2$ is optimal in $i_t^* + 2$, $y = 3$ is optimal in $i_t^* + 3$, and so on, thus proving the first claim in the proposition.

To establish concavity of $V_t(\cdot)$, we need to prove that $V_t(i+2) - V_t(i+1) \leq V_t(i+1) - V_t(i)$ for all $i \geq 0$. Consider any $i \geq 0$ and the following two possibilities: (i) $y = 0$ is optimal in i, and (ii) $y > 0$ is optimal in i. In view of the first claim in this proposition, we consider three sub-cases under the first possibility: the first, where $y = 0$ is optimal in both $i+1$ and $i+2$; the second, where $y = 0$ is optimal in $i+1$ but $y = 1$ is optimal in $i+2$; and the third, where $y = 1$ is optimal in $i+1$ and $y = 2$ is optimal in $i+2$. In the first sub-case,

$$V_t(i+2) - V_t(i+1) = U_t(i+2) - U_t(i+1) \leq U_t(i+1) - U_t(i) = V_t(i+1) - V_t(i).$$

In the second sub-case,

$$V_t(i+2) - V_t(i+1) = s + U_t(i+1) - U_t(i+1) = s \leq U_t(i+1) - U_t(i) = V_t(i+1) - V_t(i),$$

where the inequality follows because $y = 0$ is optimal in $i+1$. In the third sub-case,

$$V_t(i+2) - V_t(i+1) = 2s + U_t(i) - [s + U_t(i)] = s = s + U_t(i) - U_t(i) = V_t(i+1) - V_t(i).$$

Under the second possibility, there is only one sub-case, wherein $y+1$ is optimal in $i+1$ and $y+2$ is optimal in $i+2$. Therefore,

$$\begin{aligned} V_t(i+2) - V_t(i+1) &= \Psi_t(i+2, y+2) - \Psi_t(i+1, y+1) \\ &= 2s + sy + U_t(i-y) - [s + sy + U_t(i-y)] = s, \end{aligned}$$

which in fact equals $V_t(i+1) - V_t(i)$ because

$$\begin{aligned} V_t(i+1) - V_t(i) &= \Psi_t(i+1, y+1) - \Psi_t(i, y) \\ &= s + sy + U_t(i-y) - [sy + U_t(i-y)] = s. \end{aligned}$$

This completes the proof as all possible cases have now been covered. □

The string of results above affirms by induction that Theorem 2.1 holds.

3.1. Examples

We now analyze the three most common mechanisms studied in the literature and used in practice: multi-unit Vickrey auctions (MVA), multi-unit Dutch auctions (MDA), and Yankee auctions. In the sequel, $\psi(k;n)$ denotes the expected value of the kth largest of n iid bid random variables B with distribution $F(\cdot)$ for $k = 1, 2, \ldots, n$.

In an MVA, when the number of participating bidders n is more than the lot-size x, the x bidders with the top x bids win one item each, and the clearing-price equals the highest losing bid. Therefore, the expected revenue is given by

$$\pi(x;n) = x\psi(x+1;n), \text{ for } n > x. \tag{17}$$

In an MDA, when the number of participating bidders n is strictly more than the lot-size x, the x bidders with the top x bids win one item each, and the clearing-price equals the lowest winning bid. Therefore, the expected revenue is given by

$$\pi(x;n) = x\psi(x;n), \text{ for } n > x. \tag{18}$$

In a Yankee auction, when the number of participating bidders n is strictly more than the lot-size x, the x bidders with the top x bids win one item each by paying their own bids. Therefore, the expected revenue is defined as

$$\pi(x;n) = \sum_{k=1}^{x} \psi(k;n), \text{ for } n > x. \tag{19}$$

Analytical literature in auctions uses uniform bid distributions (see Pinker et al. [2010] for instance), and we follow this trend in our examples. Specifically, we use $B \sim \mathrm{U}[L, H]$ for some $H > L$. Discrete-uniform (denoted $N \sim \mathrm{DU}[0, M]$ for a positive integer $\hat{x}$) and Poisson (denoted $N \sim \text{Poisson}(\lambda)$ for some $\lambda > 0$) distributions are employed to model stochastic demand.

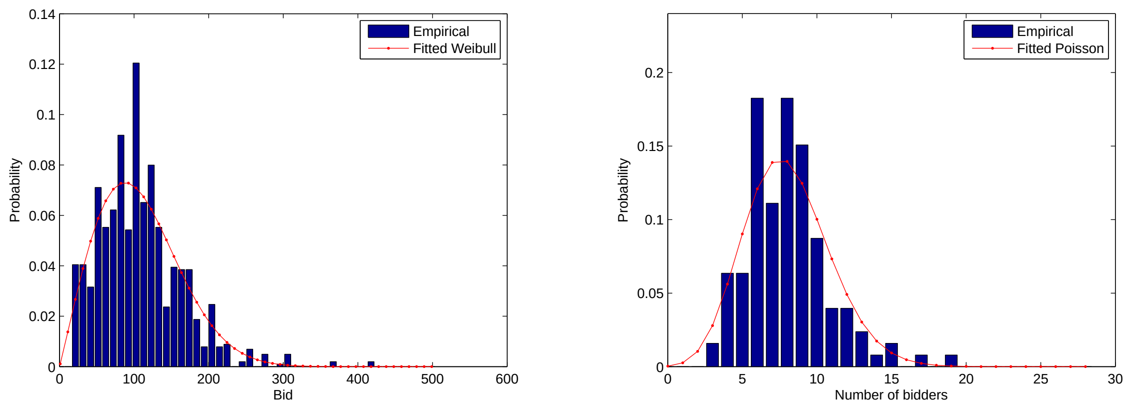
Proposition 3.12. *The second order condition (7) in Theorem 2.1 holds in MVA, MDA, and Yankee auctions with $B \sim U[L, H]$ and $N \sim DU[0, M]$ or $N \sim Poisson(\lambda)$.*

Proof. Included in Appendix Appendix A.4. □

Remark 3.13. *In Proposition 3.12, we by no means intend to claim that N and B are the same irrespective of the mechanism. Indeed, the families and/or parameters of the distributions of these random variables will likely be sensitive to the auction mechanism employed. The proposition simply lists some example combinations of mechanisms, stochastic demand distributions, and bid distributions for which condition (7) holds.*

4. Numerical experiments using real data

We collected data for 126 online auctions of a phone. It contained a total of 1158 bids ranging from \$1 to \$416. The number of bidders in one auction varied between 4 and 19. Such data typically contain some "non-serious" bids [Bapna et al. 2004; Goes et al. 2010]. We considered bids lower than 5% of the maximum to be non-serious. After removing these bids and the corresponding bidders from the data, we obtained the histograms shown in Figure 1. We fitted a Weibull distribution to the bids and a Poisson distribution[3] to the number of bidders using MATLAB®, which implements the maximum likelihood method for parameter estimation. Weibull was selected for its versatility in modeling uncertain quantities with non-negative values without a known *a priori* upper bound [Rinne 2008], and based on the shape of our bid histogram. It also provides the added benefit that an explicit formula for its expected order statistic is available. The use of Poisson random variables for modeling bidder demand is common in the empirical literature on auctions [Pinker et al. 2003; Vakrat and Seidmann 2000].



(a) (b)

Figure 1: Histograms and fitted distributions for online auction data. (a) Bids, and (b) Number of participating bidders.

Our auction data were collected from a web site that often implements MDA and hence the computations below were performed using this mechanism. We obtained the expected revenue $\pi(\cdot;\cdot)$ through formula (18). In (18), the expected value $\psi(x;n)$ of the xth largest of n iid bids from the Weibull distribution was obtained using formula 3.2.5 from Harter and Balakrishnan [1996] (also see Lieblein [1955]). This $\pi(\cdot;\cdot)$ was in turn employed to compute the single-auction expected revenue function $\phi(\cdot)$ using (1), with $L = 0$ as the bid distribution is assumed to be Weibull. The resulting expected revenue function is shown in Figure 2. We reiterate that, as hinted in Section 2, this function is not concave. We confirmed numerically that this function however did satisfy our second order sufficient condition (7).

[3] In our actual calculations below, we truncated this Poisson distribution at 150% of the largest number of participating bidders observed empirically. Such truncation is common in dynamic programming [Patrick et al. 2008] as it reduces calculations to finite sums while ignoring a negligible probability mass.

Consequently, Theorem 2.1 holds. This structural result offers significant computational savings[4] while conducting numerical experiments for sequential auctions as, for example, it assures that it suffices to compare the economic values of at most two lot-sizes for each inventory level in each auction in Bellman's backward recursion.

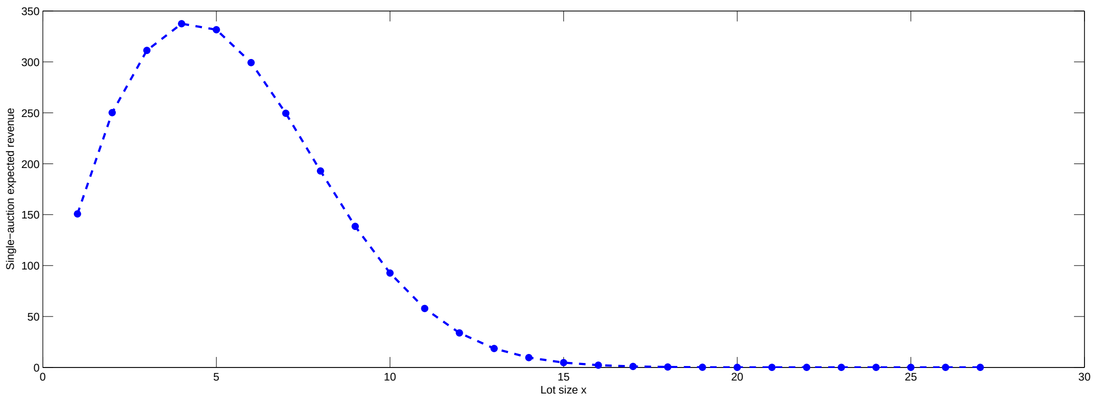
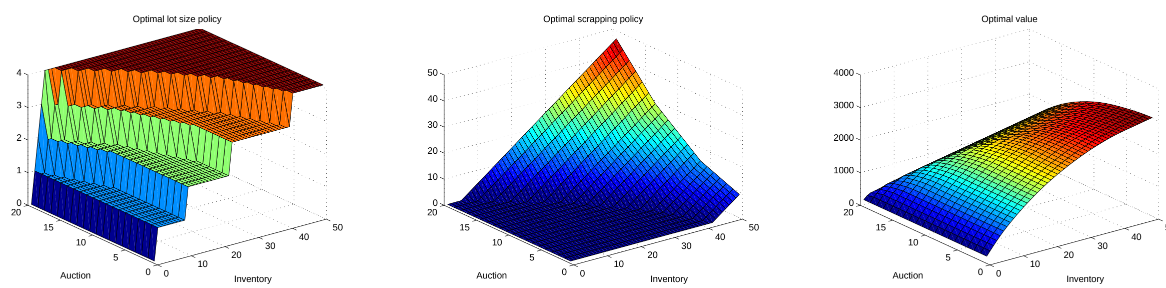


Figure 2: Single-auction expected revenue function $\phi(\cdot)$ obtained from our online auction data set.

We include here numerical results and sensitivity analyses for sequential auctions using the data above. We set initial inventory $I = 50$, discount factor $\alpha = 0.99$, and the number of auctions $T = 10$; and as the base case, unit inventory holding cost $h = 5$, and salvage value $s = 20$. Figure 3 illustrates the corresponding optimal lot-size policy, the optimal scrapping policy, and the optimal value function, all consistent in structure with the analytical results in Section 3.



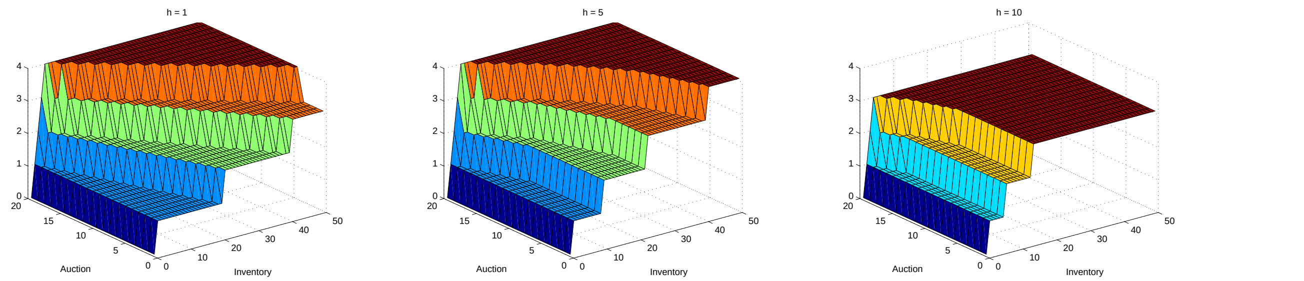
(a) (b) (c)

Figure 3: The optimal lot-sizing and scrapping policies and the optimal value function obtained from our online auction data; $I = 50$, $\alpha = 0.99$, $T = 10$, $h = 5$, and $s = 20$. (a) Optimal lot-size policy, (b) Optimal scrapping policy, and (c) Optimal value function.

For our sensitivity analyses, we experimented with different values of h and s. When we varied one parameter, the other parameter was set to its base case value given above. Figure 4 shows the optimal lot-sizes for different holding costs h. For smaller values of h, the

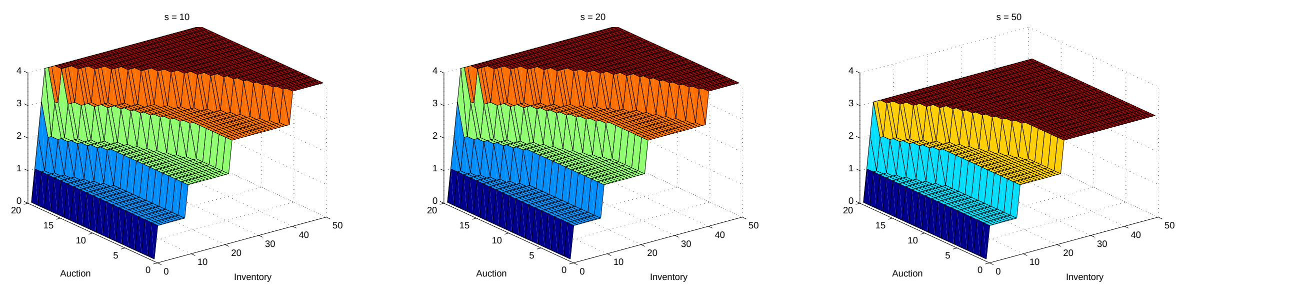
[4] For comparison, we also computed an optimal policy without exploiting our structural result, and indeed confirmed that the two policies matched.

seller holds the inventory for a longer time, i.e., auctions less in earlier auctions. Figure 5 shows the optimal lot-sizes for different salvage values s. As s increases, the optimal lot-sizes become smaller because the seller scraps more items.



(a) (b) (c)

Figure 4: Optimal lot-sizing policy for different values of unit inventory holding cost h. (a) $h = 1$, (b) $h = 5$, and (c) $h = 10$.



(a) (b) (c)

Figure 5: Optimal lot-sizing policy for different unit salvage values s. (a) $s = 10$, (b) $s = 20$, and (c) $s = 50$.

5. Discussion of assumptions, limitations, and extensions

We presented a dynamic programming model and characterized the structure of inventory scrapping and lot-sizing policies for sequential online auctions of retail goods. This model incorporates stochastic demand unlike the two existing papers in this area. This is relevant in practice because the number of participating bidders is not known *a priori* in online auctions, and the seller must plan for this uncertainty [Pinker et al. 2003]. In addition, our model allows for any auction mechanism and any bid distribution. This is helpful as the online setting offers tremendous flexibility in conducting and participating in auctions [Bapna et al. 2002, 2008; Pinker et al. 2003; Vakrat and Seidmann 2000], and our work provides a unified framework for investigating tradeoffs associated with lot-size decisions in the resulting diverse market scenarios. Some limitations and potential extensions of our model are discussed below.

A limitation of our model, as in other analytical works on online auctions [Pinker et al. 2010; Tripathi et al. 2009; van Ryzin and Vulcano 2004; Vulcano et al. 2002], is the assumption that bidders across auctions are independent. This excludes repeat bidders. Papers reviewed in the extensive survey of Pinker et al. [2003] also make the independence assumption. Pinker et al. [2003] comment that the data they gathered in Vakrat and Seidmann [2000] for hundreds of online auctions did support this assumption. Pinker et al. [2003], Pinker et al. [2010], Vulcano et al. [2002], and van Ryzin and Vulcano [2004] suggest, using slightly different arguments, that this assumption is reasonable if bidders do not "wait around" to bid in future auctions and if unsuccessful bidders are impatient and simply leave to buy elsewhere rather than participating in later auctions. One way to include repeat bidders is to assume that each bidder who loses in one auction comes back for the next auction with a certain probability, or more generally, the number of bidders across auctions evolves according to a Markov chain. Unfortunately, behavior of repeat bidders in later auctions, and in particular, the evolution of the corresponding bid distributions is not yet well-understood [Goes et al. 2010; Pinker et al. 2010]. This makes the corresponding lot-size decision problem difficult to model and solve, and we did not purse it here.

The model and the structural results in this paper do in fact allow for bid distributions $F(\cdot)$ that are parameterized by lot-size x, even though our notation suppresses this feature for simplicity. This captures bidder sensitivity to disclosed lot-sizes, for instance, where bidders shade their bids in response to higher lot-sizes [Goes et al. 2010]. In practice, it is possible that the lot-size also affects the number of participating bidders. This would call for a parameterization of the pmf $g(\cdot)$ of N by x. Our model can easily incorporate this and optimal policies can then be numerically computed; however, the structural analysis in Section 3 does not apply to this extension.

As stated in Section 2, the seller could require a minimum bid of $\Lambda > L$. This has a left-truncating effect on the support $\mathcal{B}$ of the bid distribution $F(\cdot)$ hence bumping clearing prices up, and also increases the revenue generated when an auction fails due to insufficient demand because the seller then sells items for amount Λ. On the flip side however, such a minimum bid requirement filters out some bidders thus reducing demand. This tradeoff indicates that the seller may benefit from optimizing the minimum bid requirement Λ. Unfortunately, simultaneous dynamic optimization of multiple design variables such as scrapped inventory, lot-size, and minimum bid, and especially a structural analysis of the corresponding optimal policies, will be difficult in a setting as general as that in Section 2. It may be viable under more restrictive assumptions and should provide an interesting avenue for future research.

To simplify notation, in particular, to avoid having to put a subscript t on all our random variables and other data parameters, we assumed in Section 2 that problem data do not change over auctions. However, Theorem 2.1 naturally extends even when this assumption is relaxed. In that non-stationary case, it suffices to simply require that condition (7) holds for the data in each auction. This follows by taking a closer look at our proofs to note that condition (7) only includes quantities that correspond to *one* auction and in particular that do not "mix" functions from different auctions.

Following Pinker et al. [2010] and Tripathi et al. [2009], our model assumes that all auctions are of equal duration. Again, in our model, this is purely for the sake of notational simplicity. If durations differ across auctions, the pmf $g(\cdot)$ of the number of bidders and perhaps the bid distribution $F(\cdot)$ will vary across indices t. Moreover, the cost of carrying one unit of inventory over the tth auction will depend on the duration of that auction. In fact, this variable-durations situation is one concrete example of how non-stationary data can arise, and consequently our results continue to hold there as explained in the above paragraph.

More generally, as Pinker et al. [2003] suggest, but do not themselves pursue, auction-duration for each auction could be a dynamic decision variable while designing a sequence of auctions. However, as stated earlier, a rigorous structural analysis of a model in which multiple design variables (duration, lot-size, scrapping) are dynamically optimized appears intractable in our general setting. In fact, this was not achieved even in the more restrictive, and significantly simpler settings of Pinker et al. [2010] and Tripathi et al. [2009]; and to the best of our knowledge, has never been successfully accomplished (see Etzion et al. [2006]; Odegaard and Puterman [2006]; Segev et al. [2001]; Vakrat and Seidmann [2000]; Vulcano et al. [2002]; van Ryzin and Vulcano [2004]). Part of the difficulty in meaningfully optimizing auction-durations is that, in sequential retail auctions, there is currently little empirical evidence available on exactly how auction-duration affects the pmf of the number of bidders participating in each auction and especially their bid distributions. Our model thus shares the limitation with existing literature on sequential auctions in that it does not dynamically optimize auction-durations. In the future, it may be possible, perhaps under more restrictive assumptions, to extend our model and *numerically* optimize auction-durations in sequential auctions dynamically together with the other decision variables considered in this paper. The numerical results presented in Vakrat and Seidmann [2000] and Etzion et al. [2006] on the duration of a *single* auction may provide a starting point for such an effort.

The seller can increase revenue by incorporating information acquired in early auctions into lot-size decisions in later auctions through Bayesian updates [Pinker et al. 2010]. An extension of our model that implements such a learning framework to update bid distributions as well as the pmf of stochastic demand may be possible.

Finally, the model in Section 2 assumes that the seller conducts a pre-determined, finite number of sequential auctions. Its infinite-horizon counterpart circumvents the need to predetermine the number of auctions — the sequence of auctions terminates naturally when all inventory is depleted. In this sense, the infinite-horizon model is essentially an *indefinite*-horizon model. Bellman's equations for this model are given by

$$V(i) = \max_{\substack{0 \le y \le i \\ 0 \le x \le i-y}} \Big[sy - h(i-y) + \phi(x) + \alpha E\big[V(i-y-\zeta_x)\big]\Big], \text{ for } i \ge 1, \tag{20}$$

where $V(0) = 0$. Now consider the following value iteration scheme that starts with an initial guess $V^0(i)$ for all $i \ge 1$, for the value function, and in each iteration $k \ge 1$, updates the value function from $V^{k-1}(\cdot)$ to $V^k(\cdot)$:

$$V^k(i) = \max_{\substack{0 \le y \le i \\ 0 \le x \le i-y}} \Big[sy - h(i-y) + \phi(x) + \alpha E\big[V^{k-1}(i-y-\zeta_x)\big]\Big], \text{ for } i \ge 1, \tag{21}$$

with $V^k(0) = 0$ for all $k \ge 0$. It is well-known that such a value iteration scheme converges pointwise to the optimal value function, that is, $\lim_{k\to\infty} V^k(i) = V(i)$ for all $i \ge 0$ [Puterman 1994]. By using arguments identical to Section 3 we can show that if $V^{k-1}(\cdot)$ is concave then so is $V^k(\cdot)$. Moreover, the pointwise limit preserves concavity. By choosing a concave initial guess function $V^0(\cdot)$, these ideas can be used to prove that Theorem 2.1 extends to this infinite-horizon case.

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Appendix A. Proofs of technical results

Proofs of technical results in the text are provided here.

Appendix A.1. Proof of Lemma 3.2

Let lot-sizes x_1 and x_2 be such that $x_1 \geq x_2$.

$$\bar{G}_{x_2}(w) = P(\zeta_{x_2} > w) = \begin{cases} 1 \text{ for } w < 0 \\ P(N > \lfloor w \rfloor) \text{ for } 0 \leq w < x_2 \\ 0 \text{ for } w \geq x_2, \end{cases}$$

and

$$\bar{G}_{x_1}(w) = P(\zeta_{x_1} > w) = \begin{cases} 1 \text{ for } w < 0 \\ P(N > \lfloor w \rfloor) \text{ for } 0 \leq w < x_2 \\ P(N > \lfloor w \rfloor) \text{ for } x_2 \leq w < x_1 \\ 0 \text{ for } w \geq x_1. \end{cases}$$

Thus,

$$\bar{G}_{x_1}(w) = \begin{cases} \bar{G}_{x_2}(w) \text{ for } w < x_2 \\ P(N > \lfloor w \rfloor) \geq 0 = \bar{G}_{x_2}(w) \text{ for } x_2 \leq w < x_1 \\ \bar{G}_{x_2}(w) \text{ for } w \geq x_1, \end{cases}$$

and hence $\bar{G}_{x_1}(w) \geq \bar{G}_{x_2}(w)$ for all real numbers w. Thus ζ_x is stochastically increasing.

Observe that $\min\{x, N\}$ is concave in x for every fixed N. Therefore, ζ_x is strongly stochastically concave and hence stochastically concave.

Appendix A.2. Proof of Lemma 3.8

Fix lot size x and consider j_1 and j_2 such that $j_1 \geq j_2$. Let $C_j(w) \triangleq P(\gamma_j \leq w)$ and $\bar{C}_j(w) \triangleq P(\gamma_j > w)$.

$$\bar{C}_{j_2}(w) = P(\gamma_{j_2} > w) = \begin{cases} 1 \text{ for } w < 0 \\ P(N > j_2 + \lfloor w \rfloor) \text{ for } 0 \leq w < x \\ 0 \text{ for } w \geq x, \end{cases}$$

and

$$\bar{C}_{j_1}(w) = P(\gamma_{j_1} > w) = \begin{cases} 1 \text{ for } w < 0 \\ P(N > j_1 + \lfloor w \rfloor) \text{ for } 0 \leq w < x \\ 0 \text{ for } w \geq x, \end{cases}$$

Thus,

$$\bar{C}_{j_1}(w) = \begin{cases} \bar{C}_{j_2}(w) \text{ for } w < 0 \\ P(N > j_1 + \lfloor w \rfloor) \leq P(N > j_2 + \lfloor w \rfloor) = \bar{C}_{j_2}(w) \text{ for } 0 \leq w < x \\ \bar{C}_{j_2}(w) \text{ for } w \geq x, \end{cases}$$

and hence $\bar{C}_{j_1}(w) \leq \bar{C}_{j_2}(w)$ for all real numbers w. Thus γ_j is stochastically decreasing.

$\min\{x + j, N\} - j$ is concave in j for every fixed value of N. Therefore, γ_j is strongly stochastically concave and hence stochastically concave.

Appendix A.3. Proof of Lemma 3.10

A short proof is provided below for each claim.

1. Because y is optimal in inventory i, $\Psi_t(i, y) \geq \Psi_t(i, z)$ for all $0 \leq z \leq i$. Therefore, $\Psi_t(i+1, y+1) = s + sy + U(i-y) = s + \Psi_t(i, y) \geq s + \Psi_t(i, z) = \Psi_t(i+1, z+1)$ for all $0 \leq z \leq i$. That is, $\Psi_t(i+1, y+1) \geq \Psi_t(i+1, z)$ for all $1 \leq z \leq i+1$.
2. Using $sy + U_t(i-y) \geq U_t(i)$ along with concavity of $U_t(\cdot)$ we have, for all $i' \geq i$,

$$\Psi_t(i', y) = sy + U_t(i'-y) \geq U_t(i) - U_t(i-y) + U_t(i'-y) \geq U_t(i') = \Psi_t(i', 0).$$

Appendix A.4. Proof of Proposition 3.12

To simplify notation, we set $L = 0$ and $H = 1$, and hence work with $B \sim \mathrm{U}[0, 1]$. The algebra for other values of L and H is identical. The expected value of the kth largest of n iid bids is then given by

$$\psi(k; n) = \Big(1 - \frac{k}{n+1}\Big), \text{ for } k = 1, 2, \ldots, n. \tag{A.1}$$

We include a complete proof for MVA. The proofs for MDA and Yankee are similar, and hence are omitted.

For MVA, substituting formula (A.1) into Equation (17) we get

$$\pi(x; n) = x\Big(1 - \frac{x+1}{n+1}\Big) \text{ for } n \geq x+1. \tag{A.2}$$

The single-auction expected revenue in (1) then simplifies to

$$\phi(x) = x \sum_{n=x+1}^{\infty} g(n)\Big(1 - \frac{x+1}{n+1}\Big). \tag{A.3}$$

Appendix A.4.1. $N \sim DU[0, M]$

In this case, $g(n) = 1/(M+1)$ for $n = 0, 1, \ldots, M$, and $g(n) = 0$ for $n \geq M$. Equation (A.3) then changes to

$$\phi(x) = \frac{x}{M+1} \sum_{n=x+1}^{M} \Big(1 - \frac{x+1}{n+1}\Big), \text{ for } 0 \leq x \leq M. \tag{A.4}$$

Substituting (A.4) in definition (6), after some algebraic simplification, we get

$$\partial\Phi(x) = 1 - \frac{2(x+1)}{M-x} \sum_{n=x+1}^{M} \frac{1}{n+1}, \text{ for } 0 \leq x \leq M-1. \tag{A.5}$$

Therefore,

$$\partial^2\Phi(x) = \partial\Phi(x+1) - \partial\Phi(x) = \frac{2(x+1)}{M-x} \sum_{n=x+1}^{M} \frac{1}{n+1} - \frac{2(x+2)}{M-x-1} \sum_{n=x+2}^{M} \frac{1}{n+1}, \text{ for } 0 \leq x \leq \hat{x}-2.$$

Let $\mathcal{H}(n) = 1 + 1/2 + \ldots + 1/n$ denote the n^{th} harmonic number, with the convention that $\mathcal{H}(0) = 0$. Then, for $0 \le x \le \hat{x} - 2$,

$$\partial^2 \Phi(x) = \frac{2(x+1)(\mathcal{H}(M+1) - \mathcal{H}(x+1))}{M-x} - \frac{2(x+2)(\mathcal{H}(M+1) - \mathcal{H}(x+2))}{M-x-1}. \tag{A.6}$$

Thus, $\partial^2 \Phi(x) \le 0$ if and only if $(x+1)(M-x-1)(\mathcal{H}(M+1) - \mathcal{H}(x+1)) - (x+2)(M-x)(\mathcal{H}(M+1) - \mathcal{H}(x+2)) \le 0$. After simplifying, the left hand side equals $(M-x) - (M+1)(\mathcal{H}(M+1) - \mathcal{H}(x+1))$, which is at most zero because $\mathcal{H}(\cdot)$ is a concave function whose smallest marginal value over $\{0, 1, 2, \ldots, M\}$ equals $1/(M+1)$.

Appendix A.4.2. $N \sim$ Poisson(λ)

In this case, $g(n) = \frac{e^{-\lambda}\lambda^n}{n!}$ for $n \ge 0$. Equation (A.3) then changes to

$$\phi(x) = x \sum_{n=x+1}^{\infty} \frac{e^{-\lambda}\lambda^n}{n!}\Big(1 - \frac{x+1}{n+1}\Big), \text{ for } x \ge 0. \tag{A.7}$$

This can be more compactly written as

$$\phi(x) = x(1 - G(x)) - \frac{x(x+1)(1 - G(x+1))}{\lambda}, \text{ for } x \ge 0, \tag{A.8}$$

where $G(\cdot)$ denotes the distribution function of Poisson(λ). Then $\Delta\phi(x)$ simplifies to

$$\Delta\phi(x) = \frac{2(x+1)g(x+1)}{\lambda} + (1 - G(x)) - \frac{2(x+1)(1 - G(x))}{\lambda}, \tag{A.9}$$

and hence

$$\partial\Phi(x) = \frac{\Delta\phi(x)}{1 - G(x)} = \frac{2(x+1)g(x+1)}{\lambda(1 - G(x))} + 1 - \frac{2(x+1)}{\lambda}. \tag{A.10}$$

Note that the pmf $g(\cdot)$ of a Poisson(λ) random variable satisfies $g(x) = \frac{(x+1)g(x+1)}{\lambda}$. Therefore,

$$\partial\Phi(x) = \frac{2g(x)}{(1 - G(x))} + 1 - \frac{2}{\lambda} - \frac{2x}{\lambda},$$

and hence,

$$\partial^2\Phi(x) = \partial\Phi(x+1) - \partial\Phi(x) = \frac{2g(x+1)}{(1 - G(x+1))} - \frac{2g(x)}{(1 - G(x))} - \frac{2}{\lambda}. \tag{A.11}$$

The ratio $\rho(x) \triangleq g(x)/(1 - G(x))$ is in fact the (generalized) hazard rate of the Poisson(λ) random variable N, and therefore, $\Delta\rho(x) \triangleq \frac{g(x+1)}{(1-G(x+1))} - \frac{g(x)}{(1-G(x))}$ is its first difference. We claim that the first difference $\Delta\rho(x)$ of the (generalized) hazard rate $\rho(x)$ of a Poisson(λ) random variable is at most $1/\lambda$ for all x. To prove this, we need to show that

$$\text{LHS} \triangleq \frac{\lambda g(x+1)}{1 - G(x+1)} - \frac{\lambda g(x)}{1 - G(x)} \le 1. \tag{A.12}$$

We have,

$$\begin{aligned}
\text{LHS} &= \frac{\lambda \frac{e^{-\lambda}\lambda^{x+1}}{(x+1)!}}{\frac{e^{-\lambda}\lambda^{x+2}}{(x+2)!} + \frac{e^{-\lambda}\lambda^{x+3}}{(x+3)!} + \dots} - \frac{\lambda \frac{e^{-\lambda}\lambda^{x}}{(x)!}}{\frac{e^{-\lambda}\lambda^{x+1}}{(x+1)!} + \frac{e^{-\lambda}\lambda^{x+2}}{(x+2)!} + \dots} \\
&= \frac{1}{\frac{1}{x+2} + \frac{\lambda}{(x+2)(x+3)} + \frac{\lambda^2}{(x+2)(x+3)(x+4)} + \dots} - \frac{1}{\frac{1}{x+1} + \frac{\lambda}{(x+1)(x+2)} + \dots} \\
&= \frac{1}{q(x+1)} - \frac{1}{q(x)} = \frac{q(x) - q(x+1)}{q(x)q(x+1)},
\end{aligned} \tag{A.13}$$

with $q(x) \triangleq \frac{1}{x+1} + \frac{\lambda}{(x+1)(x+2)} + \frac{\lambda^2}{(x+1)(x+2)(x+3)} + \dots$. The numerator in (A.13) is

$$\begin{aligned}
\text{LHS}_{\text{num}} &\triangleq q(x) - q(x+1) \\
&= \left(\frac{1}{x+1} - \frac{1}{x+2}\right) + \lambda\left(\frac{1}{(x+1)(x+2)} - \frac{1}{(x+2)(x+3)}\right) + \dots \\
&= \frac{1}{(x+1)(x+2)} + \frac{2\lambda}{(x+1)(x+2)(x+3)} + \frac{3\lambda^2}{(x+1)\dots(x+4)} + \dots
\end{aligned}$$

The denominator in (A.13) is

$$\begin{aligned}
\text{LHS}_{\text{den}} &\triangleq q(x)q(x+1) \\
&= \frac{1}{(x+1)(x+2)} + \frac{\lambda}{(x+1)(x+2)(x+3)} + \frac{\lambda^2}{(x+1)(x+2)(x+3)(x+4)} + \dots \\
&\quad + \frac{\lambda}{(x+1)(x+2)(x+2)} + \frac{\lambda^2}{(x+1)(x+2)(x+2)(x+3)} + \dots \\
&\quad + \frac{\lambda^2}{(x+1)(x+2)(x+3)(x+2)} + \dots,
\end{aligned}$$

which is bounded below by $\frac{1}{(x+1)(x+2)} + \frac{2\lambda}{(x+1)(x+2)(x+3)} + \frac{3\lambda^2}{(x+1)(x+2)(x+3)(x+4)} + \dots = \text{LHS}_{\text{num}}$. Therefore $\text{LHS} = \text{LHS}_{\text{num}}/\text{LHS}_{\text{den}} \leq 1$ as required in (A.12). In view of Equation (A.11), this property implies that condition (7) holds.