A Supplementary material: proofs of technical results

A.1 Proof of Lemma 3.1
For the first claim, note that \( x^* \) and \( y \) are feasible and complementary to \((P_d)\) and \((D_d)\), respectively. The result then follows from the first claim in our complementary slackness Theorem 2.2. Similarly, the second claim follows from the second claim in Theorem 2.2.

A.2 Proof of Lemma 4.1
The first claim follows because, for each \( n \), constraints (20)-(21) in \((\text{INV}(x^*))\) for \( j = 1, 2, \ldots, M_n \) and constraints (26)-(27) in \((\text{INV}^n(x^*))\) are in fact identical. This is because variables \( y_{N_n+1}, y_{N_n+2}, \ldots \) do not appear in constraints (20)-(21) for \( j = 1, 2, \ldots, M_n \). Moreover, the lower and upper bound constraints on \( d_j \), for \( j = 1, 2, \ldots, M_n \), appear both in \((\text{INV}^n(x^*))\) and in \((\text{INV}(x^*))\). These observations imply that the truncation to \( \mathbb{R}^{N_n} \times \mathbb{R}^{M_n} \) of any feasible solution to \((\text{INV}(x^*))\) is feasible to \((\text{INV}^n(x^*))\) as required for the first claim. The argument for truncations of feasible solutions to \((\text{INV}^n+1(x^*))\) as in the second claim is identical. The third claim then follows naturally.

A.3 Proof of Lemma 4.2
The objective function in \((\text{INV}^n(x^*))\) is continuous over \( \mathbb{R}^{N_n} \times \mathbb{R}^{M_n} \). The feasible region of \((\text{INV}^n(x^*))\) is closed in \( \mathbb{R}^{N_n} \times \mathbb{R}^{M_n} \) because the constraint functions are continuous. When \( C7 \) holds, this feasible region is also bounded (hence compact) and non-empty. Thus, \((\text{INV}^n(x^*))\) has an optimal solution (see Corollary 2.35 in [2]).

A.4 Proof of Lemma 4.3
The feasible regions \( F_n(x^*) \) are closed in \( Y \times Z \) for each \( n \) because the constraint functions in \((\text{INV}^n(x^*))\) are continuous. They are non-empty by assumption as in condition \( C7 \). Moreover, they belong to the compact set \( \mathcal{E} \) by condition \( C7 \). The third conclusion of Lemma 4.1 states that \( F_{n+1}(x^*) \subseteq F_n(x^*) \). Moreover, \( F(x^*) = \bigcap_{n=1}^{\infty} F_n(x^*) \). In addition, \( F(x^*) \) is a subset of the metric space \( Y \times Z \) (this space is metrizable because \( Y \) and \( Z \) themselves are metrizable — again see Theorem 3.36 on page 89 of [2]). Thus, \( F(x^*) \) is non-empty. It is also closed because it is an intersection of closed sets. Finally, it is a subset of the compact set \( \mathcal{E} \) in the metric space \( Y \times Z \). This implies that \( F(x^*) \) is compact as claimed.

A.5 Proof of Lemma 4.4
It suffices to show that the objective function is uniformly convergent over \( \mathcal{E} \) (see Theorem 9.7 in [4]). Specifically, for any \((y, d) \in \mathcal{E}\), we have that \( |d_j| \leq \delta_j \) for all \( j \). As a result, we have,

\[
\sum_{j=1}^{\infty} w_j |c_j^* - d_j| \leq \sum_{j=1}^{\infty} w_j (|c_j^*| + |d_j|) \leq \sum_{j=1}^{\infty} w_j (|c_j^*| + |\delta_j|) < \infty \text{ by (13)}. 
\]

Thus, the objective function converges uniformly over \( \mathcal{E} \) by the Weierstrass test (see Theorem 9.6 in [4]).

A.6 Proof of Corollary 4.5
Follows because the feasible region in \((\text{INV}(x^*))\) is non-empty and compact by Lemma 4.3 and the objective function in \((\text{INV}(x^*))\) is continuous by Lemma 4.4 (again see Corollary 2.35 in [2]).
A.7 Proof of Theorem 4.6

The sequence of optimal solutions \((\hat{y}(n), \hat{d}(n)) \in \mathcal{E}\) has a convergent subsequence because the set \(\mathcal{E}\) is compact. We denote this convergent subsequence by \((\hat{y}(n_k), \hat{d}(n_k))\). We use \((\bar{y}, \bar{d})\) to denote its limit. We show by contradiction that this limit is optimal to (INV\((x^*)\)). So suppose not. Suppose instead that some other solution \((\tilde{y}, \tilde{d}) \in \mathcal{E}\) is optimal to (INV\((x^*)\)) (such an optimal solution exists by Corollary 4.5). Consequently, we have,

\[
\sum_{j=1}^{\infty} w_j |c_j^* - \tilde{d}_j| < \sum_{j=1}^{\infty} w_j |c_j^* - \bar{d}_j|. \tag{54}
\]

Now we construct another subsequence \((\hat{y}(n_k), \hat{d}(n_k))\) in \(\mathcal{E}\) as follows:

\[
\hat{y}_i(n_k) = \begin{cases} 
\hat{y}_i(n_k), & i = 1, 2, \ldots, N_{n_k} \\
\tilde{y}_i, & i = N_{n_k} + 1, N_{n_k} + 2, \ldots;
\end{cases} \tag{55}
\]

and

\[
\hat{d}_j(n_k) = \begin{cases} 
\hat{d}_j(n_k), & j = 1, 2, \ldots, M_{n_k} \\
\tilde{d}_j, & j = M_{n_k} + 1, M_{n_k} + 2, \ldots.
\end{cases} \tag{56}
\]

It is important to note that, just like \((\hat{y}(n_k), \hat{d}(n_k))\), this new subsequence \((\hat{y}(n_k), \hat{d}(n_k)) \in \mathcal{E}\) also converges to the limit \((\bar{y}, \bar{d})\). As a result, the strict inequality in (54) can be rewritten as

\[
\sum_{j=1}^{\infty} w_j |c_j^* - \bar{d}_j| < \sum_{j=1}^{\infty} w_j \lim_{k \to \infty} |c_j^* - \hat{d}_j(n_k)|. \tag{57}
\]

Now, by continuity of the objective function in (INV\((x^*)\)) as shown in Lemma 4.4, the limit on the right hand side can be exchanged with the series to obtain

\[
\sum_{j=1}^{\infty} w_j |c_j^* - \hat{d}_j| < \lim_{k \to \infty} \sum_{j=1}^{\infty} w_j |c_j^* - \hat{d}_j(n_k)|. \tag{58}
\]

By splitting the series on the right hand side into two parts, this yields

\[
\sum_{j=1}^{\infty} w_j |c_j^* - \hat{d}_j| < \lim_{k \to \infty} \sum_{j=1}^{M_{n_k}} w_j |c_j^* - \hat{d}_j(n_k)| + \lim_{k \to \infty} \sum_{j=M_{n_k}+1}^{\infty} w_j |c_j^* - \hat{d}_j(n_k)|. \tag{59}
\]

Then, using the definition of the sequence \(\hat{d}(n_k)\) as in (56), the above right hand side reduces to

\[
\sum_{j=1}^{\infty} w_j |c_j^* - \hat{d}_j| < \sum_{j=1}^{M_{n_k}} w_j |c_j^* - \hat{d}_j(n_k)| + \lim_{k \to \infty} \sum_{j=M_{n_k}+1}^{\infty} w_j |c_j^* - \tilde{d}_j|. \tag{60}
\]

This implies that there is a sufficiently large \(k\), which we call \(k^*\), such that

\[
\sum_{j=1}^{\infty} w_j |c_j^* - \tilde{d}_j| < \sum_{j=1}^{M_{n_k^*}} w_j |c_j^* - \hat{d}_j(n_k^*)| + \sum_{j=M_{n_k^*}+1}^{\infty} w_j |c_j^* - \tilde{d}_j|. \tag{61}
\]
Then, splitting the left hand side series into two parts, we obtain
\[\sum_{j=1}^{M_{n_k^*}} w_j|c_j^* - \tilde{d}_j| + \sum_{j=1+M_{n_k^*}}^{\infty} w_j|c_j^* - \tilde{d}_j| < \sum_{j=1}^{M_{n_k^*}} w_j|c_j^* - \tilde{d}_j(n_{k^*})| + \sum_{j=1+M_{n_k^*}}^{\infty} w_j|c_j^* - \tilde{d}_j(n_{k^*})|. \tag{62}\]

Canceling common terms on both sides, we get
\[\sum_{j=1}^{M_{n_k^*}} w_j|c_j^* - \tilde{d}_j| < \sum_{j=1}^{M_{n_k^*}} w_j|c_j^* - \tilde{d}_j(n_{k^*})|. \tag{63}\]

But this contradicts the optimality of \((\hat{y}(n_{k^*}), \hat{d}(n_{k^*}))\) to (INV\(^{n_{k^*}}(x^*)\)) because the truncation of \((\hat{y}, \hat{d})\) to \(\mathbb{R}^{N_{n_{k^*}}} \times \mathbb{R}^{M_{n_{k^*}}}\) is feasible to (INV\(^{n_{k^*}}(x^*)\)) by Lemma 4.1. This proves the first claim.

The same argument as above holds for any convergent subsequence of \((\hat{y}(n), \hat{d}(n))\). This proves the second claim.

Lastly, to prove value convergence, we will show that every convergent subsequence of optimal objective values \(R_n(x^*)\) converges to the same limit and that this limit is \(R(x^*)\). This would imply that the sequence \(R_n(x^*)\) itself converges to \(R(x^*)\) as required. So let \(R_{n_k}(x^*)\) be such a convergent subsequence and let \((\hat{y}(n), \hat{d}(n))\) be the corresponding subsequence of optimal solutions to problems \((\text{INV}^{n_k}(x^*))\). We assume that the tail variables in \(\hat{d}(n)\) are fixed such that \(\tilde{d}_j(n_k) = c_j^*\) for \(j = 1 + M_{n_k}, 2 + M_{n_k}, \ldots\) Without loss of generality, we assume that \(\delta_j \in C7\) is large enough so that \(|c_j^*| \leq \delta_j\). Then, the subsequence \((\hat{y}(n_k), \hat{d}(n_k))\) belongs to the aforementioned compact set \(E\) and hence it has a further subsequence \((\hat{y}(n_{k^*}), \hat{d}(n_{k^*}))\) that is convergent. Then, as shown above, this subsequence must converge to a limit \((\hat{y}, \hat{d})\) that is optimal to (INV\((x^*)\)). Then, we have,

\[
\lim_{t \to \infty} R_{n_k^*}(x^*) = \lim_{t \to \infty} \sum_{j=1}^{M_{n_k^*}} w_j|c_j^* - \tilde{d}_j(n_{k^*})| \tag{64}
\]
\[
= \lim_{t \to \infty} \sum_{j=1}^{\infty} w_j|c_j^* - \tilde{d}_j(n_{k^*})| \quad \text{(because the tail of } \tilde{d}(n_{k^*}) \text{ matches that of } c^*) \tag{65}
\]
\[
= \sum_{j=1}^{\infty} w_j \lim_{t \to \infty} |c_j^* - \tilde{d}_j(n_{k^*})| \quad \text{(by continuity as shown in Lemma 4.4)} \tag{66}
\]
\[
= \sum_{j=1}^{\infty} w_j|c_j^* - \tilde{d}_j| = R(x^*). \tag{67}
\]

This shows that \(R_{n_k^*}(x^*)\) converges to \(R(x^*)\) as \(t \to \infty\). But since \(R_{n_k^*}(x^*)\) is a further subsequence of the convergent subsequence \(R_{n_k}(x^*)\), it must converge to the same limit as \(R_{n_k}(x^*)\). Consequently, \(R_{n_k}(x^*)\) also must converge to \(R(x^*)\). In summary, we have shown that every convergent subsequence of \(R_n(x^*)\) converges to \(R(x^*)\) as claimed.