4.1 LIMITS OF FUNCTIONS

Definition 1. If $x \in \mathbb{R}$, it is said to be a limit point of S if every open ball $B_\varepsilon(x)$ contains at least one point in $S \setminus \{x\}$.

Theorem 1. *The element x is a limit point of S if and only if there is a sequence $\{x_n\} \subseteq S$ such that $x_n \to x$ and $x_n \neq x$ for all $n \in \mathbb{N}$.*

Proof. $\Rightarrow$ Suppose x is a limit point of S, then there is a $y \in S$ such that $y \in B_\varepsilon(x)$ and $y \neq x$. Define a sequence $\{\varepsilon_n = 1/n\}$. Each $B_{\varepsilon_n}(x)$ contains a $y_n \in B_{\varepsilon_n}(x)$ and $y_n \neq x$, then by definition of a limit we have $y_n \to x$.

$\Leftarrow$ If $x_n \to x$, then for all $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$ for all $n \geq N$. Then for all $\varepsilon > 0$, $x_n \in S$ and $x_n \in B_\varepsilon(x)$, and by the hypothesis $x \neq x_n$, so x is a limit point. □

Now lets look at a bunch of definitions and theorems from point-set topology.

Definition 2. A point $x \in S$ is an interior point if there is a $B_\varepsilon(x) \subseteq S$; i.e. we can find a ball around x that is completely inside the set.

Definition 3. A set $S \in \mathbb{R}$ is open if it contains all its interior points.

Definition 4. A set $S \in \mathbb{R}$ is closed if $\mathbb{R} \setminus S$ is open.

Theorem 2. *A set S is closed if and only if it contains all of its limit points.*

Here are the most important definitions of this section.

Definition 5. A function $f : A \to \mathbb{R}$ has a limit L near $a \in A$ if
for all $\varepsilon > 0$ such that for all $x \in A$, $0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$.

Definition 6. A function $f : A \to \mathbb{R}$ does not have a limit L near $a \in A$ if there exists an $\varepsilon > 0$ such that for all $\delta > 0$ there is an $x \in A$ such that $0 < |x - a| < \delta \Rightarrow |f(x) - L| \geq \varepsilon$.

This is illustrated in the figure on the next page.
Now lets look at a few easy examples.

Ex: If $f(x) = x$, prove that $\lim_{x\to 1} f(x) = 1$.

Proof. Choose $\delta = \varepsilon$, then for all $\varepsilon > 0$ we have

$$|x - 1| < \delta \Rightarrow |f(x) - 1| = |x - 1| < \delta = \varepsilon.$$

□

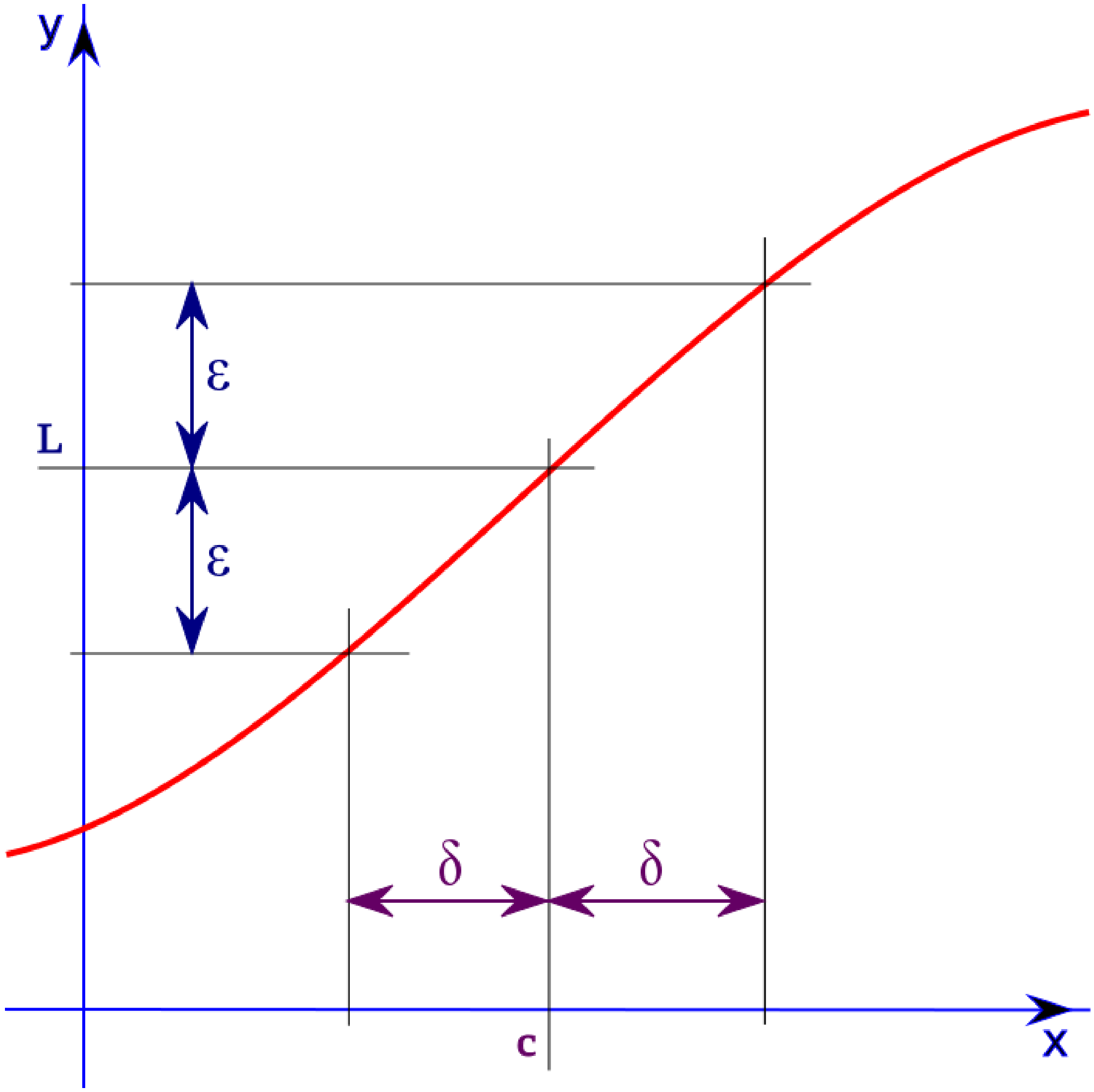
Ex: If $f(x) = x^2$, prove that $\lim_{x\to 2} f(x) = 4$.

Scratch Work: We have that $|x - 2| < \delta$ and we want $|x^2 - 4| < \varepsilon$. Notice that if $|x - 2| < 1$, then we can bound $|x+2| < 5$ because the supremum of x can be in that neighborhood is 3. Then we have that $|x^2 - 4| = |x - 2||x + 2| < 5|x - 2|$. However we need the added requirement $|x - 2| < \varepsilon/5$.

Proof. Choose $\delta = \min(1, \varepsilon/5)$, then for all $\varepsilon > 0$ we have

$$|x - 2| < \delta \Rightarrow |f(x) - 4| = |x^2 - 4| = |x - 2||x + 2| < 5|x - 2| < \varepsilon.$$

□



4.2 Limit theorems

Definition 7. Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$. Let $a \in \mathbb{R}$ be a limit point of A. We say that f is bounded near a if there is an interval $(a-\delta, a+\delta)$ such that $|f(x)| \leq M \in \mathbb{R}$ for all $x \in A \cap (a-\delta, a+\delta)$.

Theorem 3. *If $f : A \to \mathbb{R}$ has a limit at $a \in \mathbb{R}$, then f is bounded on $(a-\delta, a+\delta)$ for some $\delta > 0$.*

Proof. Since f has a limit, say L, then for all $\varepsilon > 0$, there is a $\delta > 0$ such that $0 < |x-a| < \delta \Rightarrow |f(x)-L| < \varepsilon$. Choose $\varepsilon = 1$, then

$$|f(x)| - |L| \leq |f(x) - L| < 1 \Rightarrow |f(x)| < 1 + |L|.$$

if $x \neq a$. If $x = a$, $f(x) = f(a)$. So, choose $M = max(|f(a)|, |L| + 1)$, then $|f(x)| \leq M$. □

Make sure you go over 4.2.3 - 4.2.6. Know what the sum, product, difference, and quotient of limits are.