3.4 Bolzano-Weierstrass

We already discussed subsequences before, so lets go over some properties and prove some theorems

Property 1 (Divergence criteria)**.** If $\{x_n\} \subseteq \mathbb{R}$, then it diverges if

(1) it is unbounded, or
(2) it has convergent subsequences with differing limits.

Lemma 1 (Monotone subsequences)**.** *If $\{y_n\} \subseteq \mathbb{R}$, it has a monotone subsequence.*

Here is a proof that is different from that of the book.

Proof. Define the functions

$$\begin{aligned} m(n+1) &:= \begin{cases} m(n)+1 & \text{if } y_{n+1} \geq \max\{y_1, \ldots, y_n\}, \\ m(n) & \text{otherwise;} \end{cases} \\ k(n+1) &:= \begin{cases} k(n)+1 & \text{if } y_{n+1} \leq \min\{y_1, \ldots, y_n\}, \\ k(n) & \text{otherwise;} \end{cases} \end{aligned} \tag{1}$$

Now we define $\{x_k\}$ and $\{z_m\}$ as susbsequences. Without loss of generality suppose $\{x_k\}$ terminates. If $\{z_m\}$ does not terminate, then $\{z_m\}$ is monotone (increasing) by definition. If $\{z_m\}$ also terminates and $y_n \to p$, then define

$$j(N+1) := \begin{cases} j(N)+1 & y_{N+1} \leq \min\{y_n, \ldots, y_N\} \text{ and } y_N \geq p, \\ j(N) & \text{otherwise;} \end{cases} \tag{2}$$

Notice that since we are working on z without loss of generality, we will have a similar indexing from the x-side. Since $y_n \to p$, a subsequence $\{w_j\}$ is decreasing.

If $\{y_n\}$ diverges, define a deletion set

$$D_K := \{y_n : y_n < y_K \quad \forall N < n < K\} \tag{3}$$

then $\{y_N, y_{N+1}, \ldots\} \setminus D_k$ is a decreasing sequence, thereby completing the proof. □

Theorem 1 (Bolzano–Weierstrass)**.** *If $\{x_n\} \subseteq \mathbb{R}$ is bounded, it has a convergent subsequence.*

Proof. Since it is bounded, it has a monotone subsequence that is also bounded. Further, since bounded monotone sequences converge, $\{x_n\}$ has a convergent subsequences. □

We already proved Theorem 3.4.9 earlier in the semester.

Now lets define limit superiors and inferiors.

Definition 1. Let $\{x_n\} \subseteq \mathbb{R}$ be bounded, then

(1) The limit superior of $\{x_n\}$ is the infimum of $V \subseteq \mathbb{R}$ such that $x_n > v \in V$ for at most a finite number of $n \in \mathbb{N}$, and
(2) The limit inferior of $\{x_n\}$ is the supremum of $W \subseteq \mathbb{R}$ such that $x_n > w \in W$ for at most a finite number of $n \in \mathbb{N}$.

These are denoted as $\limsup x_n$ and $\liminf x_n$. In shorthand we can write them as

$$\limsup x_n = \inf_{n\in\mathbb{N}} \left(\sup_{k\geq n} x_k \right), \quad \liminf x_n = \sup_{n\in\mathbb{N}} \left(\inf_{k\geq n} x_k \right) \tag{4}$$

Lets look at some examples of limit superior and inferior,

(1) Consider the sequence

$$\{x_n\} = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots 1 - \frac{1}{n}, \dots$$

Then

$$y_n = \sup\left\{1 - \frac{1}{k} : k \geq n\right\} = 1, \quad z_n = \inf\left\{1 - \frac{1}{k} : k \geq n\right\} = 1 - \frac{1}{n};$$

Now, $\sup y_n = 1$ and $\inf z_n = 1$ as well, so $\limsup x_n = \liminf z_n = 1$.

(2) Consider the sequence $x_n = (-1)^n$, then $\sup\{(-1)^k : k \geq n\} = 1$ and $\inf\{(-1)^k : k \geq n\} = -1$. So, $\limsup x_n = 1$ and $\liminf x_n = -1$.

(3) Consider the sequence

$$\{x_n\} = \left\{2, -\frac{3}{2}, \frac{4}{3}, -\frac{5}{4}, \cdots, \ (-1)^{n+1}\left(1 + \frac{1}{n}\right)\right\}$$

Then

$$y_n = \sup\{x_k : k \geq n\} = \begin{cases} 1 + \frac{1}{n} \text{ for } n \text{ odd,} \\ 1 + \frac{1}{n+1} \text{ for } n \text{ even;} \end{cases}$$

$$z_n = \inf\{x_k : k \geq n\} = \begin{cases} -\left(1 + \frac{1}{n+1}\right) \text{ for } n \text{ odd,} \\ -\left(1 + \frac{1}{n}\right) \text{ for } n \text{ even;} \end{cases}$$

So, $\limsup x_n = 1$ and $\liminf x_n = -1$.

(4) For the sequence $\{n\}$, $\sup\{k : k \geq n\} = \infty$ and $\inf\{k : k \geq n\} = n$, so $\limsup\{n\} = \liminf\{n\} = \infty$.

(5) Consider the sequence $\{x_n\}$ such that

$$x_n = \begin{cases} n & \text{for } n \text{ odd,} \\ \frac{1}{n} & \text{for } n \text{ even;} \end{cases}$$

then $\sup\{x_k : k \geq x_n\} = \infty$ and

$$\inf\{x_k : k \geq x_n\} = \begin{cases} n & \text{for } n \text{ odd,} \\ \frac{1}{n} & \text{for } n \text{ even;} \end{cases}$$

Therefore, $\limsup x_n = \infty$ and $\liminf x_n = 0$.

Make sure to read Theorem 3.4.11 on your own.

Theorem 2. *A bounded sequence* $\{x_n\} \subseteq \mathbb{R}$ *converges if and only if* $\limsup x_n = \liminf x_n$.

Proof. $\Rightarrow$ If $\{x_n\}$ converges, then for all $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $|x_n - p| < \epsilon$ for all $n \geq N$, so $-\epsilon < x_n - p < \epsilon$, so $p - \epsilon < x_n < p + \epsilon$. Hence, $p = \sup\{x_n\}$ and $p = \inf\{x_n\}$ for all $n \geq N$. Since this holds for all $n \geq N$, $\limsup x_n = \liminf x_n$.

$\Leftarrow$ If $\limsup x_n = \liminf x_n$, then let $y_n = \sup\{x_k : k \geq n\}$ and $z_n = \inf\{x_k : k \geq n\}$, so $z_n \leq x_n \leq y_n$ for all $n \geq N$, hence there is an x_n such that $x_n > p - \epsilon$ and $x_n < p + \epsilon$. Therefore, $p - \epsilon < x_n < p + \epsilon \Rightarrow -\epsilon < x_n - p < \epsilon \Rightarrow |x_n - p| < \epsilon$.

□

3.5 Cauchy Sequences

Lets begin with an interesting lemma,

Lemma 2. *If $\{x_n\} \subseteq \mathbb{R}$ conv., then for all $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $|x_m - x_n| < \epsilon$ for all $n, m \geq N$.*

Proof. Let $x_n \to p$, and choose N such that $|x_n - p| < \epsilon/2$ for all $n \geq N$, then $|x_m - p| < \epsilon/2$ for all $m \geq N$. Therefore, by the triangle inequality

$$|x_m - x_n| = |(x_m - p) - (x_n - p)| \leq |x_m - p| + |x_n - p| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

□

Clearly this is a special type of sequence, so lets define it.

Definition 2. A sequence $\{x_n\} \subseteq \mathbb{R}$ is called a Cauchy sequence if
for all $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $|x_m - x_n| < \epsilon$ for all $n, m \geq N$.
And this property is called the Cauchy criterion.

Basically this says that as n gets larger, the terms of the sequence get closer together.

Now lets look at a couple of examples. In class I had two different, but similarly stated examples in my head, but jumbled them up.

Ex: Consider the sequence $x_n = \frac{(-1)^{n-1}}{n}$. This will be Cauchy since

$$|x_m - x_n| = \left| \frac{(-1)^{m-1}}{m} - \frac{(-1)^{n-1}}{n} \right| \leq \left| \frac{n+m}{nm} \right| \leq \frac{2N}{N^2} = \frac{2}{N}.$$

for $n, m \geq N$, so we may choose $N > 1/2\epsilon$.

Ex: Consider the sequence $x_n = \sum_{i=1}^{n} \frac{(-1)^{i-1}}{i}$, then

$$|x_m - x_n| = \left| \sum_{i=1}^{m} \frac{(-1)^{i-1}}{i} - \sum_{i=1}^{n} \frac{(-1)^{i-1}}{i} \right| = \left| \frac{1}{n+1} - \frac{1}{n+2} + \cdots \pm \frac{1}{m} \right| < \frac{1}{n} \leq \frac{1}{N}$$

for $n < m$, without loss of generality. Then we choose $N = 1/\epsilon$, which gives us the Cauchy criterion.

Notice that Lemma 3.5.4 is obvious once we prove the next theorem. Think about why that is.

Theorem 3 (Cauchy sequences)**.** *In $\mathbb{R}$ every Cauchy sequence converges.*

Proof. Let $\{x_n\} \subseteq \mathbb{R}$ be Cauchy. If the range of $\{x_n\}$ is finite, then all except a finite number of terms are equal, and hence $\{x_n\}$ converges to this common value.

If the range is infinite, we first notice that the sequence is bounded, by the definition of a Cauchy sequence; i.e. when $\epsilon_* = 1$ there is an N such that $n \geq N \Rightarrow |x_n - x_N| < 1$. So, by Bolzano–Weierstrass $\{x_n\}$ has a convergent subsequence $\{x_m\}$ and let $x_m \to p$. Then for all $\epsilon > 0$, there is an N such that $|x_m - p| < \epsilon/2$ for all $m \geq N$. Further, since the sequence is Cauchy, we also have $|x_n - x_m| < \epsilon/2$ for all $n \geq N$. Therefore, by the triangle inequality we have

$$|x_n - p| = |(x_n - x_m) + (x_m - p)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

and hence $x_n \to p$. □

Now, this leads us to a much nicer definition of completeness.

Definition 3. A metric space S is complete if every Cauchy sequence in S converges to a point in S.

For example $\mathbb{R} \setminus \{0\}$ is not complete. Consider $\{1/n\}$. We know $1/n \to 0$, but $0 \neq \mathbb{R} \setminus \{0\}$.