2.5 Intervals

We went over examples of open, closed, and neither open nor closed intervals. We discussed what end points were and how to find the length. Recall that nested intervals are a sequence of intervals contained one inside the other; i.e. $I_n \supseteq I_{n+1}$ for all $n \in \mathbb{N}$. For example the following are both nested intervals: $I_n = [0, 1/n]$ and $I_n = (0, 1/n)$. Now, lets look at some properties of nested intervals.

Property 1 (Nested intervals)**.** If $I_n = [a_n, b_n]$ where $n \in \mathbb{N}$ is a nested sequence of closed bounded intervals, then there exists $\xi \in \mathbb{R}$ such that $\xi \in I_n$ for all $n \in \mathbb{N}$.

What happens for open sets? For example, $I_n = (0, 1/n)$. Here $I_n \to \emptyset$ since zero isn't in any I_n.

Another way we can write the above property as as follows, $\cap_{n=1}^{\infty} I_n \neq \emptyset$ and if length$(I_n) \to 0$, $\cap_{n=1}^{\infty} I_n$ contains only one point.

I will write the proof differently from that of the book, but you should be acquainted with both proofs. The more proofs you read, the better you get at proving.

Proof. Since I_n are nested, $a_1 \leq a_2 \leq \cdots \leq a_n < b_n \leq \cdots \leq b_2 \leq b_1$, so I_n is bounded for all $n \in \mathbb{N}$. Now we can define a sequence ξ_n such that $\xi_n = (a_n + b_n)/2$. Then $\xi \in I_k$ for all $k \leq n$. Since this holds for all $n \in \mathbb{N}$, $I_n \neq \emptyset$ for all $n \in \mathbb{N}$ and $\xi_n \to \xi \in I_n$ for all $n \in \mathbb{N}$. □

Make sure to read 2.5.3 and 2.5.4 on your own.

We did a few examples of binary digits and we know what decimal digits are.

Now, lets prove that $\mathbb{R}$ is uncountable. But before we jump into it, we need to convince ourselves that intervals of $\mathbb{R}$ have just as many elements as $\mathbb{R}$ itself. Consider the following function

$$f : [0, 1) \to \mathbb{R}^+, \text{ such that } f(x) := \frac{1}{1-x} - 1. \tag{1}$$

This function maps the interval into the positive reals. We would still have to show the usual properties such as bijection, so make sure you write out the bijection proof to convince yourself that this works.

Theorem 1. *$\mathbb{R}$ is uncoutnable.*

This is similar to the second uncountability proof in your book, but I change it up a bit. The strategy is going to be showing the interval $(0, 1)$ is uncountable via contradiction.

Proof. Suppose that $(0, 1)$ is countable. Then there is a sequence $s = \{s_n\}$ whose terms constitute the entire interval. Lets write $s_n = 0.u_{n,1}u_{n,2}u_{n,3}\cdots$ where each $u_{n,i} = 0, 1, \ldots, 9$. Consider the real number $y = 0.v_1v_2v_3\cdots$ where

$$v_n = \begin{cases} 1 & u_{n,n} \neq 1 \\ 2 & u_{n,n} = 1; \end{cases}$$

then there is no $\{s_n\}$ that can be y, since y differs from s_1 in $u_{1,1}$, from s_2 in $u_{2,2}$, from s_n in $u_{n,n}$, etc. □

We can use this argument because of how much "stuff" is in the reals. We can't use this argument for the rationals because every rational number either has a finite decimal representation or can be represented with a repeating decimal. So, changing one digit will just change y into a different rational number represented by some s_n. Of course, this is not a proof that the rationals are countable! This is simply a consequence of them being countable.

Before we move onto the next section lets discuss a couple of concepts that aren't in the book.

Metric spaces have the following properties for elements x, y in a metric space:

(1) $d(x, x) = 0$

(2) $d(x, y) > 0$ if $x \neq y$

(3) $d(x, y) = d(y, x)$

(4) $d(x, y) \leq d(x, z) + d(z, y)$

where d is a distance. In the reals we can use the distance $d(x, y) = |y - x|$. Notice that the last property is just the triangle inequality.

We can't give a formal definition of closure since we haven't discussed condensation points and limit points, but lets think of what it means intuitively. The closer of a set will be the set itself unioned with all the points that make it not closed. So for example the closure of $(0,1)$ is $\overline{(0,1)} = (0,1) \cup \{0,1\} = [0,1]$. The closure of the rationals is $\overline{\mathbb{Q}} = Q \cup Q^c = \mathbb{R}$. If we know what closure is, we can define density in a really nice way.

Definition 1. In a metric space M if $A \subseteq S \subseteq \overline{A}$, then A is said to be dense in S.

So, with this definition we can see that the rationals are dense in the reals. Similarly the irrationals are also dense in the reals.

3.1 Sequences

Definition 2. A sequence $\{x_n\} \subseteq \mathbb{R}$ converges if there is an $x \in \mathbb{R}$ such that
For every $\epsilon > 0$, there is an $\mathbb{N}$ such that $|x - x_n| \leq \epsilon$ for all $n \geq \mathbb{N}$;
otherwise it diverges. We call this x the limit of $\{x_n\}$.

Notation: $\lim_{n\to\infty} x_n = x$, $x_n \to x$ as $n \to \infty$, or just $x_n \to x$.

Theorem 2 (Uniqueness)**.** *A sequence $\{x_n\} \subseteq \mathbb{R}$ has at most one limit.*

Proof. Assume $x_n \to p$ and $x_n \to q$. By the triangle inequality, $|p-q| \leq |p-x_n|+|q-x_n|$. Since $|p-x_n| \to 0$ and $|q - x_n| \to 0$, $|p - q| \to 0 \Rightarrow p = q$. □

Before we do the next theorem lets discuss subsequences. For example if $s = \{1/n\}$, $k = \{2^n\}$, then $s_{k(n)} = s \circ k = \{1/2^n\}$.

Theorem 3. *In $\mathbb{R}$, $x_n \to x$ if and only if every subsequence $x_{k(n)} \to x$.*

Proof. $\Rightarrow$: Assume $x_n \to x$, then for every $\epsilon > 0$ there is an N such that $n \geq N \Rightarrow |x - x_n| < \epsilon$. Since $\{x_{k(n)}\}$ is a subsequence, there is an M such that $k(n) \geq N$ for all $n \geq M$, hence $n \geq M \Rightarrow |x - x_{k(n)}| < \epsilon$.

$\Leftarrow$: Since every subsequence $x_{k(n)} \to x$, choose $k(n) = n$, then $x_n \to x$.

□

Now lets do a few example problems for finding the limit of sequences

11) Prove that

$$\frac{1}{n} - \frac{1}{n+1} \to 0 \tag{2}$$

Solution: First lets do some back of the envelope calculations. We want to show

$$\left|\frac{1}{n} - \frac{1}{n+1}\right| < \epsilon,$$

so lets first see if we can bound our sequence by something easier to deal with

$$\left|\frac{1}{n} - \frac{1}{n+1}\right| = \left|\frac{1}{n(n+1)}\right| < \left|\frac{1}{n^2}\right| \leq \left|\frac{1}{n}\right|.$$

Now we can do the proof

Proof. For every $\epsilon > 0$, choose $N(\epsilon) = 1/\epsilon$, then for all $n \geq 1/\epsilon$,

$$\left|\frac{1}{n} - \frac{1}{n+1}\right| < \frac{1}{n} \leq \frac{1}{\epsilon} \tag{3}$$

□

15) Prove that

$$(2n)^{1/n} \to 1. \tag{4}$$

Solution: Again we do some back of the envelope calculations. Notice that

$$(2n)^{1/n} = e^{\ln(2n)/n}$$

so

$$\left|\frac{\ln(2n)}{n}\right| < \epsilon_1 \Rightarrow \left|e^{\ln(2n)/n} - 1\right| < \epsilon$$

if $\epsilon_1 = \ln(\epsilon + 1)$. Furthermore,

$$\left|\frac{\ln(2n)}{n}\right| < \frac{1}{\ln(n)} \qquad \text{for } n > 1.$$

and $1/\ln(n) < \epsilon_1$ for $n > e^{1/\epsilon_1}$. So, lets choose

$$N(\epsilon) = e^{1/\ln(\epsilon+1)}.$$

Proof. For every $\epsilon > 0$, choose $N(\epsilon) = e^{1/\ln(\epsilon+1)}$, then for all $n \geq e^{1/\ln(\epsilon+1)}$,

$$\left|\frac{\ln(2n)}{n}\right| < \frac{1}{\ln(n)} \leq \ln(\epsilon+1) \Rightarrow \left|(2n)^{1/n} - 1\right| = \left|e^{\ln(2n)/n} - 1\right| < \left|e^{\ln(\epsilon+1)} - 1\right| = \epsilon.$$

□

Ex: Prove that $r^n \to 0$ if $|r| < 1$.

Solution: Notice that $|r|^n = \exp(n \ln|r|)$, so if $n \ln|r| = \ln(\epsilon)$, then $n = \ln(\epsilon)/\ln|r|$. Clearly this n doesn't work, but it gives us a starting point.

Proof. For every $\epsilon > 0$, choose $N(\epsilon) > \ln\epsilon - \ln|r|$, then for all $n \geq N$,

$$|r^n| = \left|\exp(n \ln|r|)\right| < \left|\exp\left(\frac{\ln\epsilon}{\ln|r|}\ln|r|\right)\right| = \epsilon$$

□