6.2 THE MEAN VALUE THEOREM

Here we will just cover a bunch of theorems that will lead up to the Mean Value Theorem.

Theorem 1. *If f defined on (a,b) has a local maximum (or minimum) at x, and f is differentiable at x, then $f'(x)=0$.*

Proof. Since $f(x)$ is maximum at x, $f(x)\geq f(x+h)$ for all h such that $x+h\in(a,b)$. Then $f(x+h)-f(x)\leq 0$. If $h\geq 0$,

$$\lim_{h\to 0^+}\frac{f(x+h)-f(x)}{h}\leq 0,$$

and if $h<0$,

$$\lim_{h\to 0^-}\frac{f(x+h)-f(x)}{h}\geq 0.$$

Since the derivative exists, $f'(x)=0$ because otherwise the left hand derivative and right hand derivative would be different. □

Theorem 2 (Rolle's)**.** *If f is continuous on $[a,b]$ and differentiable on (a,b), and $f(a)=f(b)$, then there is an $x\in(a,b)$ such that $f'(x)=0$.*

Proof. Since f is continuous, a maximum and minimum exist. If the maximum or minimum occurs in the interior, $f'(x)=0$ by the previous theorem. If they occur at the end points, $f(x)=f(a)=f(b)$, so it is a constant, and therefore the derivative is trivially $f'(x)=0$. □

Theorem 3 (Mean Value Theorem)**.** *If f is continuous on $[a,b]$ and differentiable on (a,b), then there is a $\xi\in(a,b)$ such that*

$$f'(\xi)=\frac{f(b)-f(a)}{b-a}. \tag{1}$$

Proof. Lets define a function

$$h(x):=f(x)-\frac{f(b)-f(a)}{b-a}[x-a]. \tag{2}$$

Notice that h satisfies the hypotheses of Rolle's theorem. Then $h'(\xi)=0$ for some $\xi\in(a,b)$, and hence

$$f'(\xi)-\frac{f(b)-f(a)}{b-a}=0\Rightarrow f'(\xi)=\frac{f(b)-f(a)}{b-a}.$$

□

6.3 INDETERMINATE FORMS

We will just go over this briefly since you have seen all of this in Calc I.

Recall the types of indeterminate forms

$$\frac{0}{0}\qquad\frac{\infty}{\infty}\qquad 0\cdot\infty\qquad\infty-\infty\qquad 0^0\qquad 1^\infty\qquad \infty^0.$$

Remember that L'Hôpital can only be used with the first two cases, which means you would need to convert any other case to the type in the first two: $0/0$ or ∞/∞.

Lets look at a couple of examples,

$$\lim_{x\to\infty}1^x=1$$

because the base is already unity. It is not changing. So if we take x as big as we want 1^x will still be 1.

Now lets look at a 1^∞ case that is actually indeterminate,

$$L = \lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x.$$

For this one we need to use our $e^{\ln}$ trick.

$$L = \exp\left(\lim_{x\to\infty} x \ln\left(1 + \frac{1}{x}\right)\right)$$

We need to look at the argument separately, and then plug it back in if it exists. Notice that the argument, however, is not in a proper indeterminate form. We need to change it to one of the two cases where we can use L'Hôpital.

$$\lim_{x\to\infty} x \ln\left(1 + \frac{1}{x}\right) = \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{1/x}.$$

Then applying L'Hôpital give us

$$\lim_{x\to\infty} \frac{\cancel{(1/x)'}/\ln\left(1+\frac{1}{x}\right)}{\cancel{(1/x)'}} = \lim_{x\to\infty} \frac{1}{\ln\left(1+\frac{1}{x}\right)} = 1.$$

Plugging back into the original limit gives us

$$L = \exp\left(\lim_{x\to\infty} x \ln\left(1 + \frac{1}{x}\right)\right) = e$$

6.4 Taylor's Theorem

Suppose the function f has the following power series:

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n. \tag{3}$$

Can we figure out what the coefficients are? Yes, yes we can. Notice that $f(a) = c_0$, so that gives us the first coefficient. For the second one lets differentiate to get $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots$. Now, if we plug in a we get $f'(a) = c_1$. How about the third? Well, $f''(x) = 2c_2 + 6c_3(x-a) + \cdots$, so $f''(a) = 2c_2$. Can we figure out what c_n should be? Well we see that if we keep taking derivatives and evaluating them at the center, we get $f^{(n)}(x) = n!c_n + \cdots$, so $c_n = f^{(n)}(x)/n!$. We have just derived a general formula for finding the coefficients of our series.

Definition 1. The series representation

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots \tag{4}$$

is called a Taylor series of f at $x = a$. If $a = 0$ we simply call this the Taylor series of f at $x = 0$ or the McLaurin series of f - both are used interchangeably.

Theorem 4 (Taylor)**.** *Let $f : [a,b] \mapsto \mathbb{R}$ have n continuous derivatives and let $f^{(n+1)}$ exist on (a,b). Then for $x_0 \in [a,b]$,*

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2}f''(x_0)(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1} \tag{5}$$

for some ξ between x and x_0.

Proof. For some t between x and x_0 define F as

$$F(t) := f(x) - f(t) - (x-t)f'(t) - \cdots - \frac{(x-t)^n}{n!} f^{(n)}(t). \tag{6}$$

Taking the derivative gives us

$$F'(t) = -f'(t) + f'(t) - (x-t)f''(t) + \frac{2}{2!}(x-t)f''(t) - \cdots - \frac{(x-t)^{n-1}}{(n-1)!} f^{(n)}(t) + \frac{n}{n!}(x-t)^{n-1} f^{(n)}(t) - \frac{(x-t)^n}{n!} f^{(n+1)}(t).$$

Now define

$$G(t) := F(t) - \left(\frac{x-t}{x-x_0}\right)^{n+1} F(x_0), \tag{7}$$

then $G(x_0) = G(x)$. By the Mean Value Theorem, there is a ξ between x and x_0 such that $G'(\xi) = 0$. Therefore,

$$F(x_0) = -\frac{1}{n+1} \cdot \frac{(x-x_0)^{n+1}}{(x-\xi)^n} F'(\xi) = -\frac{1}{n+1} \cdot \frac{(x-x_0)^{n+1}}{(x-\xi)^n} \cdot \frac{(x-t)^n}{n!} f^{(n+1)}(t) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}.$$

Notice that this is precisely the remainder of the Taylor series. □

Ex: Find the Taylor series of $f(x) = e^x$ and its radius of convergence.

Solution: This is easy because we can find the n^{th} derivative of e^x straightaway, i.e. $f^{(n)}(x) = e^x$, hence $f^{(n)}(0) = 1$. So $e^x = \sum_{n=0}^{\infty} x^n/n!$. Now, this is still a power series so like any other power series we can find the radius of convergence by using either root or ratio test. Lets apply ratio test,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \frac{|x|}{n+1}.$$

Taking the limit gives us $\lim_{n\to\infty} |a_{n+1}/a_n| = 0$, so $R = \infty$. Therefore, the Taylor series converges everywhere and it is an exact representation of e^x.

Ex: Find the Taylor series of $f(x) = \sin x$.

Solution: Again we have a nice pattern for this one (Hint: I like functions with nice patterns!) $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f'''(0) = -1$, and the pattern just keeps repeating, so

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$