5.4 UNIFORM CONTINUITY

Definition 1. Let $A \subseteq \mathbb{R}$, and $f : A \mapsto \mathbb{R}$. We say f is uniformly continuous on A if for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x, y \in A$, $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.

Now lets look at a few examples,

Ex: $f(x) = 1/x$ is not uniformly continuous on $A = (0, 1]$

Proof. Let $\varepsilon = 10$ and suppose we can find $0 < \delta < 1$ to satisfy $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$. However, if $x = \delta$ and $y = \delta/11$ we have

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| = \frac{11}{\delta} - \frac{1}{\delta} = \frac{10}{\delta} > 10 = \varepsilon.$$

□

Ex: $f(x) = x^2$ is uniformly continuous on $A = (0, 1]$

Proof. Notice that $|f(x) - f(y)| = |x^2 - y^2| = |(x - y)(x + y)| < 2|x - y|$. If $|x - y| < \delta$, then $|f(x) - f(y)| < 2\delta$, then choosing $\delta = \varepsilon/2$ gives us $|f(x) - f(y)| < \varepsilon$. □

Ex: Now lets show that $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.

Proof. Let $\varepsilon = 2$, and suppose that we can find $\delta > 2$ to satisfy $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$. However, if $x = \delta$ and $y = \delta/2$ we have

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| = \delta^2 - \frac{\delta^2}{4} = \frac{3}{4}\delta^2 > 3 > \varepsilon.$$

□

There seems to be an interesting connection between continuity and uniform continuity, which is something Heine proved.

Theorem 1 (Heine)**.** *Let I be a closed and bounded interval and $f : I \mapsto \mathbb{R}$ is continuous on I. Then f is uniformly continuous on I.*

When is f uniformly continuous on a nonclosed interval? I have written one result down, but lets think of other possibilities. I think if f is continuous on I and $f(\bar{I})$ is closed, then f is uniformly continuous. Let us have this as an extra credit. So prove or disprove for an extra 25 points on your homework raw score.

Theorem 2 (Continuous Extensions)**.** *A function f is uniformly continuous on (a, b) if and only if it can be defined at the end points a and b such that the extended function is continuous on $[a, b]$.*

I personally think a result on the closure would be nicer.

We already showed that a Lipschitz function is continuous, but is it uniformly continuous?

Theorem 3. *If $f : A \mapsto \mathbb{R}$ is Lipschitz continuous, then f is uniformly continuous on A*

Proof. Since f is Lipschitz, $|f(x) - f(y)| \leq K|x - y|$ for some $K > 0$. Then choose $\delta = \varepsilon/K$. Therefore,

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| \leq K|x - y| < K \cdot \frac{\varepsilon}{K} = \varepsilon.$$

□

There is also an interesting result connecting Cauchy sequences to uniformly continuous functions.

Theorem 4. *If $f : A \mapsto \mathbb{R}$ is uniformly continuous on A, then $\{x_n\}$ is Cauchy on A implies $\{f(x_n)\}$ is Cauchy on $\mathbb{R}$.*

Proof. Since $\{x_n\}$ is Cauchy, for all $\delta > 0$ there is an M such that $|x_n - x_m| < \delta$ for all $n, m \geq M$. Further, since f is uniformly continuous, for all $\varepsilon > 0$ there is a $\delta > 0$ such that $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$ for all $x, y \in A$. Therefore, $|f(x_n) - f(x_m)| < \varepsilon$ for all $n, m \geq M$, hence it is Cauchy on $\mathbb{R}$. □

Now, for the most remarkable theorem in this chapter, that we will never use.

Theorem 5 (Weierstrass Approximation Theorem)**.** *Let $f : [a, b] \mapsto \mathbb{R}$ be continuous on $[a, b]$. If $\varepsilon > 0$, then there is a polynomial p_ε such that $|f(x) - p_\varepsilon(x)| < \varepsilon$ for all $x \in [a, b]$.*

This says that the set of polynomials is dense in the set of continuous functions; i.e., any continuous function can be approximated as close as we want with a polynomial. The tricky part is finding such a polynomial. Notice that this is more powerful than Taylor's theorem because Taylor requires n derivatives, but Taylor is more useful in applications.

5.6 Monotone and Inverse Functions

We know what these are, so lets discuss some results.

Theorem 6 (Continuous Inverse)**.** *Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \mapsto \mathbb{R}$ be strictly monotone and continuous on I. Then f^{-1} is strictly monotone and continuous on $f(I)$.*

Lets also talk about jump discontinuities.

Definition 2. We say the jump of f at c to be $j_f(x) := \lim_{x\to c^+} f - \lim_{x\to c^-} f$ if $f : I \mapsto \mathbb{R}$ is increasing on I and if c is not an endpoint of I.

Theorem 7. *Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \mapsto \mathbb{R}$ be monotone on I. Then the set of points $D \subseteq I$ at which f is discontinous is a countable set.*