5.2 Combinations of continuous functions

Theorem 1. *Let $A \subseteq \mathbb{R}$, $b \in \mathbb{R}$, and $f, g : A \mapsto \mathbb{R}$. Suppose $c \in A$ and that f and g are continuous at c. Then,*

(1) *$f \pm g$, fg, and $b \cdot f$ are continuous at c, and*
(2) *if $h : A \mapsto \mathbb{R}$ is continuous at $c \in A$ and if $h(x) \neq 0$ for all $x \in A$, then f/h is also continuous at c.*

Theorem 2. *Let $A \subseteq \mathbb{R}$ and $f : A \mapsto \mathbb{R}$, then if f is continuous, so is $|f|$.*

Proof. The limit of the absolute value will equal the limit of the function itself; i.e.,

$$\lim_{x\to c} |f(x)| = |\lim_{x\to c} f(x)| = |f(c)|.$$

□

It should be noted that this special treatment does not extend to C^1 (continuous derivative) functions. Take $f(x) = x$ for example. It has a continuous derivative at $x = 0$, but $g(x) = |x|$ does not.

Theorem 3. *Let $A \subseteq \mathbb{R}$ and $f : A \mapsto \mathbb{R}$, then if f is continuous, so is $\sqrt{f}$.*

Proof. Similarly to the theorem above,

$$\lim_{x\to c} \sqrt{f(x)} = \sqrt{\lim_{x\to c} f(x)} = \sqrt{f(c)}.$$

□

Now lets prove that compositions of continuous functions is continuous.

Theorem 4. *Let $A, B \subseteq \mathbb{R}$, $f : A \mapsto \mathbb{R}$, and $g : A \mapsto \mathbb{R}$, such that $f(A) \subseteq B$. If f is continuous at $c \in A$ and g is continuous at $b = f(c) \in B$, then $g \circ f : A \mapsto \mathbb{R}$ is continuous at c.*

Proof. Since g is continuous at $b \in B$, given $\varepsilon > 0$ there exists a $\delta_1 > 0$ such that $|y-b| < \delta_1 \Rightarrow |g(y)-g(b)| < \varepsilon$. For this δ_1, since f is continuous at c, there is a δ_2 such that $|x-c| < \delta_2 \Rightarrow |f(x)-f(c)| = |y-b| < \delta_1$ if we let $y = f(x)$. Then we have that for all $\varepsilon > 0$, there is a $\delta_2 > 0$ such that $|x-c| < \delta_2 \Rightarrow |g(f(x))-g(f(c))| = |g(y) - g(b)| < \varepsilon$. □

5.3 Continuous functions on an interval

Theorem 5 (Boundedness)**.** *Suppose $f : [a, b] \mapsto \mathbb{R}$ is continuous on $[a, b]$, then f is bounded on $[a, b]$.*

Note: This does not hold for not-closed intervals. Consider the function $f(x) = 1/x$ on $(0, 1]$. It is continuous but not bounded on that interval.

Proof. Suppose f is unbounded; i.e., for any $M > 0$, $|f| > M$ for at least one $x \in [a, b]$. So, assume for $c \in [a, b]$, $|f(c)| > M$. Then for all $\delta > 0$, if $\varepsilon = M + \min(f(x))$ such that $x \in (x - \delta, x + \delta)$, $|x - c| < \delta \Rightarrow |f(x) - f(c)| > \varepsilon$. □

Theorem 6 (Max-Min)**.** *Suppose $f : [a, b] \mapsto \mathbb{R}$ is continuous on $[a, b]$, then f attains its absolute maximum and absolute minimum on $[a, b]$.*

Theorem 7 (Almost IVT)**.** *Suppose $f : [a, b] \mapsto \mathbb{R}$ is continuous on $[a, b]$, then if $f(a) < 0 < f(b)$ (or $f(a) > 0 > f(b)$), there exists a $c \in (a, b)$ such that $f(c) = 0$.*

The proof for this is quite involved, but you should read it in the book.

Theorem 8 (Intermediate Value Theorem)**.** *Suppose $f : [a, b] \mapsto \mathbb{R}$ is continuous on $[a, b]$, then if $f(a) < K < f(b)$, there is a $c \in (a, b)$ such that $f(c) = K$.*

Proof. By the previous theorem, if $f(a)-K < 0 < f(b)-K$, then there is a $c \in (a, b)$ such that $f(c)-K = 0$. Therefore if $f(a) < k < f(b)$, then $f(c) = K$. □

Corollary 1. *Suppose* $f : [\alpha, \beta] \mapsto \mathbb{R}$ *is continuous on* $[\alpha, \beta]$*. Let* $K \in \mathbb{R}$ *satisfy* $\inf(f([\alpha, \beta])) \leq K \leq \sup(f([\alpha, \beta]))$*, then there is a* $c \in [\alpha, \beta]$ *such that* $f(c) = K$*.*

Proof. By Max-Min, there exists $a, b \in [\alpha, \beta]$ such that $a = \inf(f([\alpha, \beta]))$ and $b = \sup(f([\alpha, \beta])) \Rightarrow f(c) \in [a, b]$. Then by IVT, there is a $c \in [\alpha, \beta]$ such that $f(c) = K$. □

Theorem 9. *Suppose* $f : [a, b] \mapsto \mathbb{R}$ *is continuous on* $[a, b]$*, then the set* $f([a, b]) := \{f(x) : x \in [a, b]\}$ *is a closed and bounded interval.*

Proof. Since f is continuous it is bounded, and so is $f([a, b])$.

Clearly any point $y \in f([a, b])$ such that $\inf(f([a, b])) < y < \sup(f([a, b]))$ is a limit point. Since we can choose a ball of radius $\varepsilon_* = \min(\frac{1}{2}(y - \inf(f([a, b]))), \frac{1}{2}(\sup(f([a, b]) - y)))$. Then all $B_\varepsilon(y)$ contains at least one element in $f([a, b]) \setminus \{y\}$.

Next, we see what happens if y is the supermum or infimum of the interval, which is guaranteed by the Min-Max theorem. Since $[a, b]$ is closed, f attains its max and min, which means $\sup(f([a, b])) \in f([a, b])$ and $\inf(f([a, b])) \in f([a, b])$. This would in fact prove that $f([a, b])$ is closed by a previous result, but since that was done a few months ago, lets go ahead and prove that some $y = \inf(f([a, b]))$ or $y = \sup(f([a, b]))$ is a limit point of $f([a, b])$. Notice that by the definition of the supremum (and similarly infimum), $y - \varepsilon \in f([a, b])$, then all balls $B_\varepsilon(y)$ contains elements in $f([a, b]) \setminus \{y\}$. Since $f([a, b])$ contains all of its limit points, it is closed. □

Corollary 2. *Suppose* $f : I \mapsto \mathbb{R}$ *is continuous and* I *is an interval, then the set* $f(I)$ *is also an interval.*

Now lets look at a couple of important problems from the book.

5.3.5) Here we must show the polynomial $p(x)$ has at least two real roots.

Proof. Since $p(0) = -9$ and $p(-8) = 503$, by IVT the function $p(x) = 0$ for some $x \in (-8, 0)$. Further, since $p(0) = -9$ and $p(2) = 63$, by IVT the function $p(x) = 0$ for some $x \in (0, 2)$. Therefore, it has at least two roots. □

5.3.6) For this problem we need to prove that $f(c) = f(c + 1/2)$ for some $c \in [0, 1/2]$ if $f(0) = f(1)$.

Proof. Consider the function $g(x) = f(x) - f(x + 1/2)$. Without loss of generality, assume that $f(0) > f(1/2)$. Then $g(0) = f(0) - f(1/2) > 0$ and $g(1/2) = f(1/2) - f(1) = f(1/2) - f(0) < 0$. Therefore, by IVT $g(x) = 0$ at some point $x = c \in [0, 1/2]$. □