Most important things to know: Proving the existence of a limit, continuity, uniform continuity, and Lipschitz continuity. The following definitions and theorems are in order of relevance to the exam questions.

Definition 1. A function $f : A \to \mathbb{R}$ has a limit L near $a \in A$ if
for all $\varepsilon > 0$ such that for all $x \in A$, $0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$.

Definition 2. The function f is continuous at $x = a$ if $\lim_{x\to a} f(x) = f(a)$;
i.e., for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all x $0 < |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$;
otherwise f is said to be discontinuous at $x = a$.

Notice that if $\lim_{x\to a} f(x) \neq f(a)$, then f is discontinuous at $x = a$ even if the limit exists.

Definition 3. Let $A \subseteq \mathbb{R}$, and $f : A \mapsto \mathbb{R}$. We say f is uniformly continuous on A if
for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x, y \in A$, $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.

Definition 4. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be Lipschitz continuous if $|f(x) - f(y)| \leq K|x - y|$ for all $x, y \in \mathbb{R}$, where $K > 0$.

Theorem 1 (Intermediate Value Theorem)**.** *Suppose $f : [a, b] \mapsto \mathbb{R}$ is continuous on $[a, b]$, then if $f(a) < K < f(b)$, there is a $c \in (a, b)$ such that $f(c) = K$.*

Theorem 2. *If $f : A \mapsto \mathbb{R}$ is Lipschitz continuous, then f is uniformly continuous on A*

Proof. Since f is Lipschitz, $|f(x) - f(y)| \leq K|x - y|$ for some $K > 0$. Then choose $\delta = \varepsilon/K$. Therefore,

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| \leq K|x - y| < K \cdot \frac{\varepsilon}{K} = \varepsilon.$$

□

Theorem 3 (Max-Min)**.** *Suppose $f : [a, b] \mapsto \mathbb{R}$ is continuous on $[a, b]$, then f attains its absolute maximum and absolute minimum on $[a, b]$.*

Theorem 4. *Let $f : A \to \mathbb{R}$. If $\lim_{x\to a} f(x) > 0$, then for $(a - \delta, a + \delta)$, $f(x) > 0$ for some $\delta > 0$ and all $x \in A \cap (a - \delta, a + \delta) \setminus \{a\}$.*

Definition 5. Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, and $a \in \mathbb{R}$ be a limit point of A. Then

(1) We say $f \to \infty$ as $x \to a$; i.e., $\lim_{x\to a} f(x) = \infty$, if for all $M \in \mathbb{R}$ there is a $\delta(M) > 0$ such that for all $x \in A$, $0 < |x - a| < \delta \Rightarrow f(x) > M$.

(2) We say $f \to -\infty$ as $x \to a$; i.e., $\lim_{x\to a} f(x) = -\infty$, if for all $m \in \mathbb{R}$ there is a $\delta(m) > 0$ such that for all $x \in A$, $0 < |x - a| < \delta \Rightarrow f(x) > m$.

Definition 6. Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$. Suppose that $(a, \infty) \subseteq A$ for some $a \in \mathbb{R}$. We say $L \in \mathbb{R}$ is a limit of f as $x \to \infty$; i.e., $\lim_{x\to\infty} f(x) = L$, if for all $\varepsilon > 0$ there is a $K(\varepsilon) > a$ such that for all $x > K$, $|f(x) - L| < \varepsilon$.

The exam is scored as follows: 25 limits, 30 continuity, 15 uniform continuity, 15 Lipschitz continuity, 15 Intermediate Value Theorem and basic logic.

Note: I will never ask you to prove a theorem. I don't want you memorizing proofs!