9.1 Relations

There are many ways to relate numbers or even different types of elements. We will be mainly concerned with functional relations. So let us restrict ourselves to $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$. Consider $a \in A$ and $b \in B$, and a relation $R = \{(x,2),(y,1),(y,2)\}$, then a is related to b (denoted aRb) if $(a,b) \in R$.

We also say that R is a relation from set A to B. Then the domain and range of R is defined as

$$\text{dom}(R) := \{a \in A : (a,b) \in R \text{ for some } b \in B,$$
$$\text{range}(R) := \{b \in B : (a,b) \in R \text{ for some } a \in A.$$

Further, the inverse relation is defined as

$$R^{-1} := \{(b,a) : (a,b) \in R\}.$$

We did a bunch of exercises for this section from the book.

9.2 Properties of relations

The book has some nice examples for these, which you should read up on, but lets list the properties here.

- Reflexive: if xRx for all $x \in A$; i.e., R is reflexive if $(x,x) \in R$.
- Symmetric: $xRy \Rightarrow yRx$ for all $x,y \in A$.
- Transitive: If xRy and yRz, then xRz for all $x,y,z \in A$.

Now lets look at an example.

Ex: Consider the distance between real numbers as $|x-y|$. Further, consider the relation xRy if $|x-y| \leq 1$. Lets see which properties hold for this relation.
- Reflexive (xRx): Yes, since $|x-x| = 0 \leq 1$ for all $x \in \mathbb{R}$.
- Symmetric ($xRy \Rightarrow yRx$): Yes, since $|x-y| = |y-x|$, so if $|x-y| \leq 1$, then so is $|y-x|$.
- Transitive ($xRy\textbf{AND}yRz \Rightarrow xRz$): No, due to the counterexample $x=3, y=2, z=1$.

Now lets look at some exercises from the book.

9.11) Not symmetric because $(a,b) \in R$, but $(b,a) \neq R$.

9.13) Not reflexive because $(a,a),(b,b),(c,c) \notin R$ and not symmetric because $(a,b) \in R$, but $(b,a) \notin R$.

9.18) We did this one in class, but there was a bit of ambiguity about the definitions, so let me not put it here.

9.3 Equivalence relations

A relation R is said to be an equivalence relation if R is reflexive, symmetric, and transitive. An example of this is the following: for $A = \{1,2,3,4,5,6\}$

$$R = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(1,3),(1,6),(6,1),(6,3),(3,1),(3,6),(2,4),(4,2)\}$$

For an equivalence relation R of A, the equivalence class of $a \in A$ is defined as $[a] = \{x \in A : xRa\}$.

Lets look at some examples that were in the book, and then do an exercise.

Ex: From the previous example, $[1] = \{1,3,6\}$, $[2] = \{2,4\}$, $[3] = \{1,3,6\}$, $[4] = \{2,4\}$, $[5] = \{5\}$, and $[6] = \{1,3,6\}$. Since $[1] = [3] = [6]$ and $[2] = [4]$, there are only three distinct equivalence classes, namely $[1],[2],[5]$.

Ex: Consider the equals relation on $\mathbb{Z}$; i.e., aRb defined as $a = b$. Then

$$[a] := \{x \in \mathbb{Z} : xRa\} = \{x \in \mathbb{Z} : x = a\} = \{a\}.$$

Ex: For xRy defined as $|x| = |y|$ on $\mathbb{Z}$, we have $[a] = \{-a, a\}$ if $a \neq 0$ and $[0] = \{0\}$.

9.24) Here $A = \{a, b, \ldots, g\}$, and we are given a few terms of the relation, then the complete relation is

$$R = \{(a,c),(c,d),(d,g),(b,f),(a,a),(b,b),(c,c),(d,d),(e,e),(f,f),(g,g),(c,a),(d,c),(g,d),(f,b),\\ (a,d),(d,a),(c,g),(a,g),(g,a),(g,c)\}.$$

Further, the equivalence classes are $[a] = \{a, c, d, g\}$, $[b] = \{b, f\}$, $[c] = a$, $[d] = [a]$, $[e] = \{e\}$, $[f] = [b]$, $[g] = [a]$.

9.4 Properties of equivalence classes

The book has a list of these properties presented as theorems. You should read these, but we will mainly be concerned with two properties:

- $[a] = [b] \Leftrightarrow aRb$,
- The set of distinct equivalence classes resulting from R is a partition (subsets with no overlap) of A.

Now lets do a couple of exercise problems.

9.36) The equivalence relation will be

$$R = \{(v,v),(w,w),(x,x),(y,y),(z,z),(v,w),(w,x),(v,x),(w,v),(x,w),(x,v)\}$$

Notice that we can leave two elements aside to be the distinct classes and then connect the other three.

9.40) Here we prove that this is an equivalence relation.

Reflexive (xRx): Notice that $3x - 7x = -4x = 2(-2x)$, which is even.

Symmetric ($xRy \Rightarrow yRx$): Suppose that $3x - 7y = 2m$, then

$$7x - 3y = 3x - 7y + (4x + 4y) = 2m + (4x + 4y) = 2(m + 2x + 2y),$$

which is even.

Transitive ($xRy\textbf{AND}yRz \Rightarrow xRz$): Suppose that $3x - 7y = 2m$ and $3y - 7z = 2n$ then

$$3x - 7z = (3x - 7y) + (3y - 7z) + 4y = 2m + 2n + 4y = 2(m + n + 2y),$$

which is even.

9.5 Congruence modulo n

Suppose there is a $c \in \mathbb{Z}$ such that $b = ac$ for $a, b, n \in \mathbb{Z}$ and $n \geq 2$, then a is said to be congruent to b modulo n, denoted as $a \equiv b \mod n$. Recall that $\mod$ just resets the counter, so a congruence occurs when two numbers are equivalent after reset. We looked at a bunch of examples of this in class. The examples we will do for this will be similar to the even and odd proofs.

Now lets look at some exercises

9.44) (a) True
(b) False
(c) True
(d) False

9.45) Again we want to prove this is an equivalence relation.

Reflexive (xRx): Notice that $3a + 5a = 8a \equiv 0 \mod 8$ because $a \in \mathbb{Z}$ and $8a$ is divisible by 8.

Symmetric ($xRy \Rightarrow yRx$): Suppose $3a + 5b \equiv 0 \mod 8$; i.e., $3a + 5b = 8k$. Then

$$5a + 3b = (8a + 8b) - (3a + 5b) = 8(a + b - k),$$

which is divisible by 8.

Transitive ($xRy\textbf{AND}yRz \Rightarrow xRz$): Suppose that $3a + 5b \equiv 0 \mod 8$ and $3b + 5c \equiv 0 \mod 8$; i.e., $3a + 5b = 8k$ and $3b + 5c = 8m$, then

$$(3a + 5b) + (3b + 5c) = (3a + 5c) + 8b = 8k + 8m \Rightarrow 3a + 5c = 8(k + m - b),$$

which is divisible by 8.

9.46) If $a = 1$, $a + a = 2 \neq 0 \mod 3$, so it is not reflexive.

9.47) There are two distinct equivalence classes: $[0] = \{0, \pm 2, \pm 4, \ldots\}$ and $[1] = \{\pm 1, \pm 3, \pm 5, \ldots\}$.

9.6 Integers modulo n

This is not something I know much about, so we will cover it only briefly.

Notice that the equivalence classes of $a \equiv b \mod n$ are $[0], [1], \ldots, [n-1]$. Denote these equivalence classes as $\mathbb{Z}_n$ and we call them integers modulo n.

We call operators, say addition or multiplication, operations on a set S if $x + y \in S$ and equivalently $xy \in S$. And if $T \subset S$ where $T \neq \emptyset$, then for all $x, y \in T$ we have $x + y \in S$ and $xy \in S$. The set T is said to be closed under addition or multiplication if $x + y \in T$ and equivalently $xy \in T$.

For example, the even integers are closed under both addition and multiplication, but the odd integers are only closed under multiplication not addition.

The book shows an example of adding and subtracting on $\mathbb{Z}_n$, but is this always possible? If the sum/product of two equivalence classes $[a], [b] \in \mathbb{Z}_n$ do not depend on the representatives a, b, then the operator is said to be well-defined. For $\mathbb{Z}_n$ specifically, addition and multiplication in $\mathbb{Z}_n$ are well-defined if $[a] = [b]$ and $[c] = [d]$, then $[a + c] = [b + d]$ and equivalently $[ac] = [bc]$.