A proof is derived from accepted truths, called axioms, through the use of logical operations.

3.1 Trivial and vacuous proofs

If in the implication $P \Rightarrow Q$, Q is always true, then we have a trivial proof. For example, "Let $n \in \mathbb{Z}$. If n^3, then 3 is odd." This is trivial because 3 is odd regardless of $n^3 > 0$. If, on the other hand, P is always false, then we have a vacuous proof. For example, "Let $n \in \mathbb{Z}$. If 3 is even, then $n^3 > 0$." This is vacuous because 3 is never even.

Lets look at two more examples:

Ex: "Let $x \in \mathbb{R}$. If $x < 0$, then $x^2 + 1 > 0$." is trivial since $x^2 + 1$ is always positive.
Ex: "Let $x \in \mathbb{R}$. If $x^2 - 2x + 2 \leq 0$, then $x^3 \geq 8$." is vacuous since $x^2 - 2x + 2 > 0$ for all x.

Since these cases are to be avoided, for the exercises let us simply state whether it is trivial or vacuous instead of "proving".

3.1) $x^2 - 2x + 2 \neq 0$ is always true so this is trivial.
3.3) $(r^2 + 1)/r = r + 1/r$. If $r > 1$, $r + 1/r > 1$. If $r < 1$, $1/r > 1 \Rightarrow r + 1/r > 1$. If $r = 1$, $(r^2 + 1)/r = 2 > 1$. So it is vacuous.
3.5) If $n = 1$, $n + 1/n = 2$. If $n \geq 2$, $n + 1/n > 2$. So it is vacuous.

3.2 Direct proofs

In direct proofs we go from a true statement to the conclusion through direct implications. The best way to learn proofs is to do and read them, so for every section you should read the book examples in addition to notes.

Now lets work on a bunch of exercises.

3.8) *Proof.* Since x is odd and 9 is odd, $9x$ is odd. Further, since 5 is also odd $9x + 5$ is even by the sum off odd integers. □

3.10) *Proof.* Since a and c are odd, ab and bc will either be even or odd. Then since the sum of even numbers is even and the sum of odd numbers is odd, $ab + bc$ is odd. □

3.12) *Proof.* This is only odd when $x = 0$ (then $2^{2x} = 1$), and hence $2^{-2x} = 1$ is also odd. □

3.14) This is vacuous because for every n in that set the quantity is even.

3.3 Contrapositive

As we observed earlier, $\bar{Q} \Rightarrow \bar{P}$, called the contrapositive of $P \Rightarrow Q$, is equivalent to $P \Rightarrow Q$.
Here is an example of a contrapositive

- If 3 is odd, then 57 is prime. (original)
- If 57 is not prime, then 3 is even. (contrapositive)

Sometimes it will be easier to prove the contrapositive. Lets look at some exercise problems for this.

3.16) *Proof.* Since $7x + 5$ is even, we may rewrite this as $7x + 5 = 2m + 1$ where $m \in \mathbb{Z}$. Then $7x = 2m - 4 = 2(m - 2)$, which is even. Since $7x$ is even and 7 is odd, then x must be even. □

3.18) For this problem we have a bi-conditional, so we need to prove both directions.

Proof. $\Rightarrow$ First we prove if $5x - 11$ is odd, then x is odd. Notice that $5x - 11 = 2m$ for $m \in \mathbb{Z}$, then $5x = 2m + 11 = 2(m + 5) + 1$, which is odd, x must be odd as an even value for x would produce an even result.

$\Leftarrow$ Next we prove if x is odd, then $5x - 11$ is even. The product of odd integers is odd, so $5x$ is odd. And the sum of odd integers is even, which proves the statement.

□

3.20) *Proof.* Notice that $5x - 2 = (3x + 1) + [2(x - 2) + 1]$, and $2(x - 2) + 1$ is odd.

$\Rightarrow$ If $3x + 1$ is even, then by summing an even and an odd integer $5x - 2$ is odd.

$\Leftarrow$ If $5x - 2$ is odd, $3x + 1 = (5x - 2) - [2(x - 2) + 1]$ is even since the sum of two odd integers is even.

□

3.4 Proof by cases

We discussed a little bit about cases before. Often when we look at cases, the arguments for two cases may be the same, so we use the term W.L.O.G (without loss of generality) when we need only one argument.

Lets look at some exercises where we look at different cases.

3.26) *Proof.* First suppose that n is odd; then since n^2 is odd, and so is $3n$, $n^2 - 3n + 9$ is odd by the product and sum of three odd integers. Next suppose n is even, then n^2 and $3n$ are even, and hence so is $n^2 - 3n$. Since the sum of an even and an odd integer is odd, $n^2 - 3n + 9$ is odd.

□

3.28) *Proof.* Consider the contrapositive: if x, y are even, then so is xy. This is true by the product of even integers, and hence if xy is odd, x, y are odd. □

3.30) *Proof.* $\Rightarrow$ Suppose x, y are both even, then $x = 2m$, $y = 2n$ for $n, m \in \mathbb{Z}$. So, $x - y = 2m - 2n = 2(m-n)$. Suppose x, y are both odd, then $x = 2m+1$, $y = 2n+1$. So, $x-y = (2m+1)-(2n+1) = 2(m - n)$.

$\Leftarrow$ Suppose $x - y = 2m$, then $x = 2m + y$. If y is even (W.L.O.G), then so is x and if y is odd (W.L.O.G), then so is x, hence x, y have the same parity.

□