Before we begin let us do an exercise problem from the previous section.

1.41a) Here they want a distinct collection of n set that have an intersection of zero and a union of the interval $[0,1]$. This is more of a recognition problem where you either see it or you don't. We will notice that $A_n = [0,1/n]$ gives us $\cap_{n=1}^{\infty} A_n = \{0\}$ and $\cup_{n=1}^{\infty} A_n = [0,1]$ because $1/n \to 0$ as $n \to \infty$ and $[0,1/n]$ are nested intervals where $[0,1]$ is the interval that encompasses all the others.

2.1 Truth values

For this section it is perhaps best to look at some exercise problems.

2.1) Here they want us to indicate which ones are statements and their truth values.
- (a) This is a statement, however it is **false**.
- (b) This is a **true** statement.
- (c) This is not a statement, but rather a question.
- (d) This is what's called an open statement. It has different truth values as you change the variable x, so it doesn't have a specific truth value.
- (e) This is not a statement, it is a command.
- (f) This is another open statement.

2.3) Here they want us to simply state which ones are true ore false and provide a reasoning.
- (a) False because $\emptyset$ has no elements.
- (b) True.
- (c) True.
- (d) False because $\emptyset$ has no elements.
- (e) True.
- (f) False because the element 1 is not a set.

2.4) Here we have an open statement and we need to state for which values the statement is true/false.
- (a) It is true for $x = -2$ or $x = 3$.
- (b) It is false for $x \neq -2, 3$.

2.2 Negations

The simplest logical operator is the negation operator. This just negates, or states the opposite, of the statement. Sometimes the negation is true and sometimes it is false. This all depends on the truth value of the original statement. Let us look at a couple of exercise problems.

2.11) Here they just want us to state the negation of the statements given.
- (a) $\sqrt{2}$ is an irrational number.
- (b) 0 is a negative integer.
- (c) 111 is not a prime.

2.12) Here they want us to complete the truth table.

P	Q	$\bar{P}$	$\bar{Q}$
T	T	F	F
T	F	F	T
F	T	T	F
F	F	T	T

2.3 OR and AND

With these operators we can combine statements. It should be noted that I am using slightly different notation for the sake of understandability. In the book an **OR** operation is called disjunction and an **AND** operation is called a conjunction.

For the **OR** operator only one part needs to be true for the entire statement to be true. Lets look at the following examples.

- 1 is a number **OR** a is a letter is true because both statements are true.
- 1 is a number **OR** a is a number is true because one statement is true.
- 1 is a letter **OR** a is a number is false because both statements are false.

We can also write out a truth table for this

P	Q	P **OR** Q
T	T	T
T	F	T
F	T	T
F	F	F

For the **AND** operator all parts need to be true for the entire statement to be true. Lets look at the following examples.

- 1 is a number **AND** a is a letter is true because both statements are true.
- 1 is a number **AND** a is a number is false because one statement is false.
- 1 is a letter **AND** a is a number is false because both statements are false.

We can also write out a truth table for this

P	Q	P **AND** Q
T	T	T
T	F	F
F	T	F
F	F	F

Now lets look at a couple of exercises

2.15) Here they want us to complete the truth table.

P	Q	$\bar{Q}$	P **AND** $\bar{Q}$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

2.16) Here they want us to decide which statements are true, but first let us determine the truth values of P and Q. Notice that A is not a subset of B since it contains elements that are not in B, so P is false. However, Q is true because we showed Card$(A \setminus B) = 6$. Then the true statements are a, c, d, and e.

2.4 Implications

An implication is a statement where "If P, then Q" or equivalently, "P implies Q" also written as $P \Rightarrow Q$.

The truth table for this is

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

This may seem odd, but lets consider the following examples

- "If x is divisible by 2, then x is even." is true if the first statement is true because both statements are true
- "If x is divisible by 2, then x is odd." is false if the first statement is true because while the first statement is true the second is false, and a true statement cannot imply a false one.
- Now consider a case where the first statement is false, say for $x = 3$. Then "If x is divisible by 2, then x is odd." is trivially true (also called a vacuous statement) because 3 is in fact odd regardless of what the first statement says. Further, "If x is divisible by 2, then x is even." is true because if 3 was divisible by 2 by definition it would be even.

Lets look at some exercises for more examples.

2.19) Here they want us to write each statement in words and then state whether it is true or not.

(a) "17 is odd". True.
(b) "17 is even or 19 is prime." True.
(c) "17 is even and 19 is prime" False.
(d) "If 17 is even, then 19 is prime. True.

2.20) Here they want us to write the truth table for $(P \Rightarrow Q) \Rightarrow \bar{P}$.

P	Q	$P \Rightarrow Q$	$\bar{P}$	$(P \Rightarrow Q) \Rightarrow \bar{P}$
T	T	T	F	F
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

2.5 Implications (continued)

Since we are continuing implications lets just look at a few examples.

2.30) Here they want us to state the following in words and for the closed statements to state the truth value.
(a) If 5n+3 is prime, then 7n+1 is prime.
(b) "If 13 is prime, then 15 is prime." False.
(c) "If 33 is prime, then 43 is prime." True.

2.31) Here they want us to determine for which values of x the following statements are true.
(a) x = -3, 1, 4, 5
(b) All of them.
(c) Homework

2.32) Again we determine for which x values the following statements are true.
(a) $x \neq 7$ because $7 - 3 = 4$, but $7 < 8$.
(b) Homework
(c) For all $x \in \mathbb{N}$.
(d) Homework

2.33) Again we do the same thing as the previous two
(a) (x,y) = (3,4) and (5,5).
(b) Homework
(c) Homework

2.6 Biconditional

For statements (or open statements), $Q \Rightarrow P$ is called the converse of $P \Rightarrow Q$, and $P \Leftrightarrow Q$ is called the biconditional of P and Q. In terms of logical operations we can think of $P \Leftrightarrow Q$ to be $P \Rightarrow Q$ **AND** $Q \Rightarrow P$.

From this we can construct the truth table.

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$P \Leftrightarrow Q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

The biconditional is also often states as:

- P if and only if Q,
- P is equivallent to Q,
- P is necessary and sufficient for Q.

Lets look at an example.

Ex: (a) "2 is even if and only if 3 is odd" is true because both are true.
(b) "3 is even if and only if 2 is odd" is true because both are false.
(c) "2 is odd if and only if 4 is even" is false because one is false.

Now lets look at some exercises in the book.

2.35) "18 is odd if and only if 25 is even". This is true because both are false.

2.39) For this they want us to state which values of x the following are true for.

(a) $|x| = 4 \Leftrightarrow x = 4$ is true for $x = -3, 1, 4, 5$.

(b) $x \geq 3 \Leftrightarrow 4x - 1 > 12$ is true for $x = 0, 2, 4, 6$.

(c) $x^2 = 16 \Leftrightarrow x^2 - 4x = 0$ is true for $x = -6, 3, 4, 8$

2.7 Tautologies and contradictions

A tautology is something that is trivially true. For example, "A closed statement is always true or false" is always true no matter what the statement is. An example of this is described by the following truth table:

P	$\bar{P}$	$P\,\mathbf{OR}\,\bar{P}$
T	F	T
F	T	T

Another example, $\bar{Q}\,\mathbf{OR}\,(P \Rightarrow Q)$ is a tautology as described by the following truth table:

P	Q	$\bar{Q}$	$P \Rightarrow Q$	$\bar{Q}\,\mathbf{OR}\,(P \Rightarrow Q)$
T	T	F	T	T
T	F	T	F	T
F	T	F	T	T
F	F	T	T	T

A contradiction is something that is trivially false. For example, "A closed statement is true and false" is always false no matter what the statement is. An example of this is described by the following truth table:

P	$\bar{P}$	$P\,\mathbf{AND}\,\bar{P}$
T	F	F
F	T	F

Another example, $(P\,\mathbf{AND}\,Q)\,\mathbf{AND}\,(Q \Rightarrow \bar{P})$ is a contradiction as described by the following truth table:

P	Q	$\bar{P}$	$P\,\mathbf{AND}\,Q$	$Q \Rightarrow \bar{P}$	$(P\,\mathbf{AND}\,Q)\,\mathbf{AND}\,(Q \Rightarrow \bar{P})$
T	T	F	T	F	F
T	F	F	F	T	F
F	T	T	F	T	F
F	F	T	F	T	F

Now lets look at a couple of more examples, that are in the book, so I will simply construct the truth table here.

Ex:

P	Q	$P\,\mathbf{AND}\,Q$	$P\,\mathbf{OR}\,Q$	$(P\,\mathbf{AND}\,Q) \Rightarrow (P\,\mathbf{OR}\,Q)$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

Ex:

P	Q	$\bar{P}$	$P\,\mathbf{AND}\,Q$	$Q \Rightarrow \bar{P}$	$(P\,\mathbf{AND}\,Q)\,\mathbf{AND}\,(Q \Rightarrow \bar{P})$
T	T	F	F	T	F
T	F	T	T	F	F
F	T	F	F	T	F
F	F	T	F	T	F

2.8 Logical equivalence

We looked at a logical equivalence previously where $\overline{\bar{P}\,\mathbf{AND}\,\bar{Q}} \equiv P\,\mathbf{AND}\,Q$. If you were not in class you missed it. But lets look at some simpler examples: $(P \Rightarrow Q) \equiv (\bar{P}\,\mathbf{OR}\,Q)$ and $(P\,\mathbf{AND}\,Q) \equiv (Q\,\mathbf{AND}\,P)$.

These have the following two truth tables respectively:

P	Q	$\bar{P}$	$P \Rightarrow Q$	$\bar{P}\,\mathbf{OR}\,Q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Now lets look at a couple of exercises.

2.53) For this problem they want us the construct the truth table.

(a)

P	Q	$\bar{P}$	$\bar{Q}$	$\bar{P} \Rightarrow \bar{Q}$	$P \Rightarrow Q$
T	T	F	F	T	T
T	F	T	T	T	F
F	T	F	F	F	T
F	F	T	T	T	T

As we can see, the inverse is not an equivalence of $P \Rightarrow Q$,

but as we will see in later sections, the contraposiitve is:

$P \Rightarrow Q$	$\bar{Q} \Rightarrow \bar{P}$
T	T
F	F
T	T
T	T

2.54) Here for part a and b we can construct one big truth table

P	Q	$\overline{P \textbf{ OR } Q}$	$\bar{P} \textbf{ OR } \bar{Q}$	$\overline{P \textbf{ OR } Q} \Leftrightarrow \bar{P} \textbf{OR} \bar{Q}$
T	T	F	F	T
T	F	F	T	F
F	T	F	T	F
F	F	T	T	T