CH. 4 LAPLACE TRANSFORMS

Laplace Transforms: Differential equations are hard! With the characteristic polynomial we were able to convert the problem into an algebraic equation, but this only works for simple problems. For harder problems there are special types of transforms called integral transforms.

Definition 1. If $f(t)$ is defined for all $t > 0$ and if $s \in \mathbb{R}$ such that the integral

$$F(S) = \int_0^\infty e^{-st} f(t) dt \tag{1}$$

converges for $s < s_n < \infty$, then $F(s)$ is called the Laplace Transform of $f(t)$ and denoted as $Ł\{f(t)\} = F(s)$.

Now lets do some examples

Ex: $f(t) = e^{at} \sinh bt$

Solution: Here we use the definition of the Laplace Transform

$$\begin{aligned}
F(s) &= \int_0^\infty e^{-st} e^{at} \sinh(bt) dt = \int_0^\infty e^{-st} e^{at} \frac{1}{2}(e^{bt} - e^{-bt}) dt = \frac{1}{2} \int_0^\infty \left[e^{(b+a-s)t} - e^{(-b+a-s)t} \right] \\
&= \lim_{\tau \to \infty} \frac{1}{2} \int_0^\tau \left[e^{(b+a-s)t} - e^{(-b+a-s)t} \right] = \frac{1}{2} \lim_{\tau \to \infty} \left[\frac{1}{b+a-s} e^{(b+a-s)t} - \frac{1}{-b+a-s} e^{(-b+a-s)t} \right]_0^\tau \\
&= \frac{1}{2} \lim_{\tau \to \infty} \left[\frac{1}{b+a-s} e^{(b+a-s)\tau} - \frac{1}{-b+a-s} e^{(-b+a-s)\tau} - \frac{1}{b+a-s} - \frac{1}{-b+a-s} \right] \\
&= \frac{1}{2} \left[\frac{1}{s-a-b} - \frac{1}{s-a+b} \right] = \frac{b}{(s-a)^2 - b^2}.
\end{aligned}$$

However, notice that we can only take the integral if both $b + a - s < 0$ and $-b + a - s < 0$, which means we need $s - a > |b|$.

Ex: $f(t) = e^{at} \cos bt$

Solution: Again we use the definition

$$\begin{aligned}
F(s) &= \int_0^\infty e^{-st} e^{at} \cos bt = \int_0^\infty e^{-st} e^{at} \frac{1}{2}(e^{ibt} - e^{-ibt}) dt = \frac{1}{2} \int_0^\infty \left(e^{[(a+ib)-s]t} - e^{[(a-ib)-s]t} \right) dt \\
&= \lim_{\tau \to \infty} \frac{1}{2} \int_0^\tau \left(e^{[(a+ib)-s]t} - e^{[(a-ib)-s]t} \right) dt = \frac{1}{2} \lim_{\tau \to \infty} \left[\frac{1}{(a+ib)-s} e^{[(a+ib)-s]t} - \frac{1}{(a-ib)-s} e^{[(a-ib)-s]t} \right]_0^\tau \\
&= \frac{1}{2} \lim_{\tau \to \infty} \left[\frac{1}{(a+ib)-s} e^{[(a+ib)-s]\tau} - \frac{1}{(a-ib)-s} e^{[(a-ib)-s]\tau} - \frac{1}{(a+ib)-s} + \frac{1}{(a-ib)-s} \right] \\
&= \frac{1}{2} \left[\frac{1}{(a-ib)-s} - \frac{1}{(a+ib)-s} \right] = \frac{s-a}{(s-a)^2 + b^2}.
\end{aligned}$$

The condition for this is more difficult, but if we ignore the complex part, which we can do because of certain properties of complex numbers, we get that the integral converges for $s > a$.

Ex:

$$f(t) = \begin{cases} t, & 0 \le t < 1 \\ 0, & 1 \le t < \infty \end{cases}$$

Solution:

This one is much easier than the previous two. Notice that after $t = 1$ the function is zero, so our integral becomes

$$F(s) = \int_0^\infty e^{-st} f(t) dt = \int_0^1 t e^{-st} dt = -\frac{1}{s^2} e^{-st}(st+1) \Big|_0^1 = -\frac{1}{s^2} e^{-s}(s+1) + \frac{1}{s^2}.$$

IVPs with Laplace Transforms: I'll "handwave" this section because deeper knowledge is required to properly understand the theory, which exceeds the scope of this course.

In order to apply Laplace transforms to ODEs we have to take the Laplace of the derivatives. Let $Y = \text{Ł}\{y\}$ and $y' = dy/dt$, then

$$\text{Ł}\{y'\} = \int_0^\infty e^{-st}y'dt = e^{-st}y\Big|_0^\infty + s\int_0^\infty e^{-st}ydt = -y(0) + sY. \tag{2}$$

It should be noted that this integral was done using integration by parts. We can get higher derivatives by induction

$$\text{Ł}\{y''\} = \text{Ł}\{(y')'\} = -y'(0) + s\text{Ł}\{y'\} = -y'(0) - sy(0) + s^2Y. \tag{3}$$

Lets do a few examples

Ex: Find the inverse Laplace transform of

$$F(s) = \frac{2s+2}{s^2+2s+5}$$

Solution: We first recognize what it resembles and try to convert it into that form

$$\frac{2s+2}{s^2+2s+5} = 2\frac{s+1}{(s+1)^2+4} \Rightarrow \text{Ł}^{-1}\{F(s)\} = 2e^{-t}\cos 2t.$$

Ex:

$$F(s) = \frac{8s^2-4s+12}{s(s^2+4)}.$$

Solution: First we do the partial fractions

$$\frac{A}{s} + \frac{Bs+C}{s^2+4} \Rightarrow As^2 + 4A + Bs^2 + Cs = (A+B)s^2 + Cs + 4A = 8s^2 - 4s + 12.$$

Then we get $A = 3$, $C = -4$, and $B = 5$, then

$$F(s) = \frac{3}{s} + 5\frac{s}{s^2+4} - 2\frac{2}{s^2+4} \Rightarrow \text{Ł}\{F(s)\} = 3 + 5\cos 2t - 2\sin 2t.$$

Ex: $y'' - 4y' + 4y = 0$; $y(0) = y'(0) = 1$.

Solution: We take the Laplace transform of the entire ODE

$$-\cancelto{1}{y'(0)} - s\cancelto{1}{y(0)} + s^2Y + 4\cancelto{1}{y(0)} - 4sY + 4Y = 0 \Rightarrow (s^2-4s+4)Y = s-3 \Rightarrow Y = \frac{s-3}{s^2-4s+4}$$

$$\Rightarrow Y = \frac{s-3}{(s-2)^2} = \frac{\cancel{s-2}}{(s-2)^{\cancel{2}}} - \frac{1}{(s-2)^2} \Rightarrow y(t) = e^{2t} - te^{2t}.$$

Ex: $y'' - 2y' + 2y = e^{-t}$; $y(0) = 0$, $y'(0) = 1$.

Solution: Again we take the Laplace transform of the entire ODE

$$-\cancelto{1}{y'(0)} - s\cancelto{0}{y(0)} + s^2Y + 2\cancelto{0}{y(0)} - 2sY + 2Y = \frac{1}{s+1} \Rightarrow (s^2-2s+2)Y = \frac{1}{s+1} + 1$$

$$\Rightarrow Y = \frac{1}{(s+1)(s^2-2s+2)} + \frac{1}{s^2-2s+2}.$$

The second term is fine, but for the first time we must do partial fractions, which you'll have to use a lot

$$\frac{A}{s+1} + \frac{Bs+C}{s^2-2s+2} \Rightarrow As^2 - 2As + 2A + Bs^2 + Bs + Cs + C = (A+B)s^2 + (B+C-2A)s + 2A + C = 1.$$

Then we get $A = -B \Rightarrow C = 3A$, then $A = 1/5 = -B$, and $C = 3/5$. Hence,

$$Y = \frac{1}{5}\cdot\frac{1}{s+1} + \frac{1}{5}\cdot\frac{-s+8}{s^2-2s+2} = \frac{1}{5}\cdot\frac{1}{s+1} - \frac{1}{5}\cdot\frac{s-1}{(s-1)^2+1} + 7\cdot\frac{1}{(s-1)^2+1}$$

$$\Rightarrow y = \frac{1}{5}[e^{-t} - e^t\cos t + 7e^t\sin t].$$