1.3 and 2.1.1 Basic Models and Direction Fields

What is a Differential Equation and why do we study them?

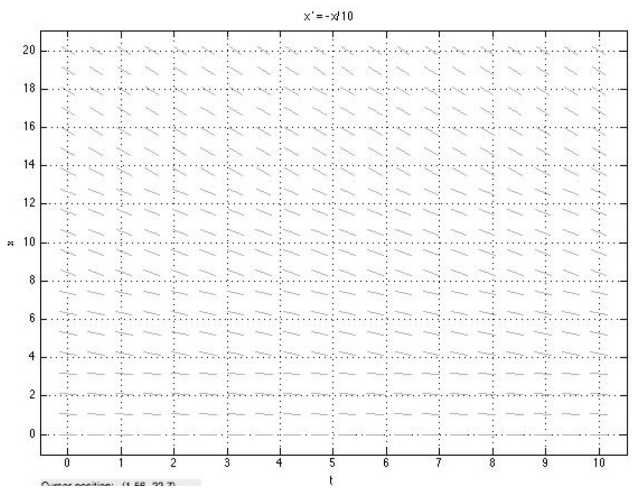
Many things in life ranging from nuclear physics to love affairs involves changes in on quantity in relation to another. Since we understand rates of change as "differentials" it is natural to model these phenomena as differential equations.

Ex: Consider carbon dating. We know that all living things contain C_{12} and C_{14}, however when living things die the C_{14} starts to decay because it is radioactive. We can model this decay. First let t be the time that has elapsed since the death of the body. Also, let $x(t)$ be the amount of C_{14} left after a period of time t. We know the decay is linear, so our rate of change will be governed by the following equation,

$$\frac{dx}{dt} = -kx,$$

where k is the rate constant. Basically the C_{14} decay at a rate of kx [mass/time].

We also discussed direction fields for this problem, which is given bellow. For the field first calculate $dx/dt = 0$, called the "equilibrium solution" or "fixed point", which is the easiest solution to find. Then after plotting that slope for the respective x, t values find what the other slopes are for other relevant x and t values.



The next example is a very important example that appears on exams frequently.

Ex: A pond initially contains 1,000,000 gal of water and an unknown amount of an undesirable chemical. Water containing 0.01 g of this chemical per gallon flows into the pond at a rate of 300 gal/h. The mixture flows out at the same rate, so the amount of water in the pond remains constant. Assume that the chemical is uniformly distributed throughout the pond.

(a) Write a differential equation for the amount of chemical in the pond at any time.

(b) How much of the chemical will be in the pond after a very long time? Does this limiting amount depend on the amount that was present initially?

Solution:

a) We want to find the rate at which the amount of chemical is changing. When we develop models one should always keep track of the dimensions. It provides you with information and reduces the chance of making a mistake. Now, let $x(t)$ be the amount of chemical in grams at a time t in hours. We know that the rate will be dx/dt, but to find the model we must realize that Total Rate = (Rate in) - (Rate out). Notice that (Rate in) = (.01 g/gal) × (300 gal/h) = 3 g/h, and (Rate out) = (300 gal/h) × (x/(1 Million) g/gal) = (3/1000)x g/h. So, our model becomes,

$$\frac{dx}{dt} = 3 - (3 \times 10^{-4})x.$$

b) What are they asking for this problem? They want to know if the amount of the chemical blows up or converges to something. To do this we look for an equilibrium solution,

$$\frac{dx}{dt} = 3 - (3 \times 10^{-4})x = 0 \Rightarrow x = 10^4.$$

For the next couple of problems they want us to find the ODE that gives us the behavior delineated in the problem.

In each of Problems 7 through 10, write down a differential equation of the form $dy/dt = ay + b$ whose solutions have the required behavior as $t \to \infty$.

7. All solutions approach $y = 3$.
8. All solutions approach $y = 2/3$.
9. All other solutions diverge from $y = 2$.
10. All other solutions diverge from $y = 1/3$.

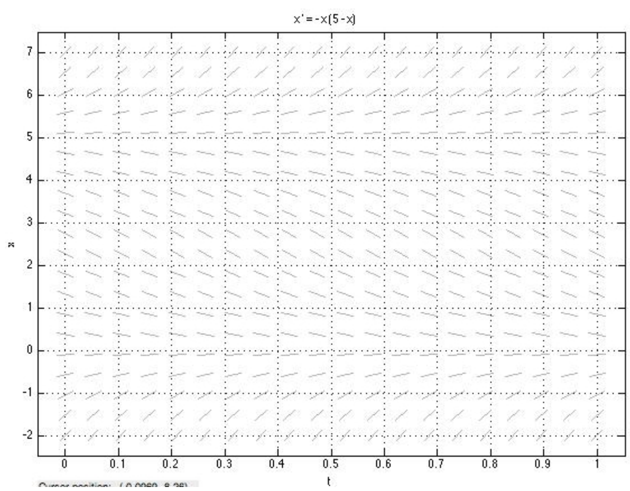
In each of Problems 11 through 14, draw a direction field for the given differential equation. Based on the direction field, determine the behavior of y as $t \to \infty$. If this behavior depends on the initial value of y at $t = 0$, describe this dependency. Note that in these problems the equations are not of the form $y' = ay + b$, and the behavior of their solutions is somewhat more complicated than for the equations in the text.

11. $y' = y(4 - y)$
12. $y' = -y(5 - y)$
13. $y' = y^2$
14. $y' = y(y - 2)^2$

7) $a \cdot 3 + b = 0 \Rightarrow b = -3a$, so in general the equation will be $y' = -ay + 3a$ since the solutions "approach".
9) $a \cdot 2 + b = 0 \Rightarrow b = -2a$, so in general the equation will be $y' = ay - 2a$ since the solutions "diverge from".

The next problem is to find the direction field.

12) We first find the equilibrium solutions of $y' = -y(5 - y)$, which are $y_* = 0$ and $y_* = 5$. To find the "stability" of the solutions, which means whether or not the solution diverges or converges to an equilibrium solution, we employ the first derivative test. This gives the following direction field,



In class we also did problems 9 and 10 on page 27:

9) This is similar to the example above where we want to find Total Rate = (Rate in) - (Rate out). As we showed in class (Rate in) = (0 g/gal) × (3 gal/min) = 0, and (Rate out) = (x/300 g/gal) × (3 gal/min) = x/100 g/min. Then

$$\frac{dx}{dt} = \frac{x}{100}; \quad x(0) = 50. \tag{1}$$

10) Since is involves a nonautonomous ODE we will look at it some other time.