## 3.7: Applications: Without Forcing

Consider a mass on a weightless-hanging spring. Gravity balances with the spring force, so we can neglect it. All we need are the additional forces on the system. The total force on the entire system is $mx''$. The spring force is $kx$, and the retarding (or damping) force is $\gamma x'$. Any external force is neglected in this section, but if it were not it would just be $F(t)$. Applying Newton's laws gives,

$$mx'' = -kx - \gamma x' \Rightarrow mx'' + \gamma x' + kx = 0;\ x(0) = x_0,\ x'(0) = v_0. \tag{1}$$

We also have to be aware of the units: $m = [\text{mass}]$, $\gamma = [\text{mass/time}]$, and $k = [\text{mass/time}^2]$.

We can separate the possible solutions into a few cases:

**Undamped:** Here $\gamma = 0$, so our equation becomes

$$mx'' + kx = 0. \tag{2}$$

And the solution is

$$x = A\cos\sqrt{\frac{k}{m}}t + B\sin\sqrt{\frac{k}{m}}t. \tag{3}$$

Here $\omega = \sqrt{k/m}$ is called the natural frequency. Now, lets think of this in the complex plane and try to determine some important quantities. We can do this by switching to the complex formulation and letting $\cos\omega t$ be the x-axis and $\sin\omega t$ be the y-axis. Then we can draw a triangle where $A$ is the base and $B$ is the height. Also, let the angle adjacent to the x-axis be called $\phi$. Then we have that the hypotenuse, $R = \sqrt{A^2 + B^2}$ is the amplitude of oscillation, and the angle $\phi = \tan^{-1}(B/A)$ is the phase. Then $A = R\cos\phi$ and $B = R\sin\phi$. Then using trig identities we get

$$x = R\cos\phi\cos\omega t + R\sin\phi\sin\omega t = R\cos(\omega t - \phi). \tag{4}$$

Now, notice that $x(0) = R\cos\phi$. When does $x = R\cos\phi$ again in the same manner? We can show this happens at every addition of $2\pi/\omega$, so our period is $T = 2\pi/\omega = 2\pi\sqrt{m/k}$.

**Damped:** Now we explore what happens when we have damping. This gives rise to three cases. Here we will have the full ODE, so our roots of the characteristic polynomial are

$$r = \frac{1}{2m}(-\gamma \pm \sqrt{\gamma^2 - 4mk}).$$

If $\gamma^2 - 4mk > 0$, our solution becomes $x = c_1e^{r_1t} + c_2e^{r_2t}$, and this is called overdamped, because it goes to zero very fast.

If $\gamma^2 - 4mk = 0$, our solution becomes $x = (c_1 + c_2t)e^{rt}$, where $r$ is a repeated root, and this is called critically damped because after some critical point it damps to zero very fast.

If $\gamma^2 - 4mk < 0$, our solution becomes $x = e^{\xi t}(A\cos\theta t + B\sin\theta t)$. This is a bit of a special case. If $\xi = -\gamma/2m$ is large, it acts like the preceding case, if it is small then we get behavior called underdamped motion. This is because the system will oscillate while damping out. Here $\theta$ is called the quasi frequency and $\theta/\omega < 1$. Similarly, $T_d = 2\pi/\theta$ is called the quasi period and $T_d/T > 1$. Also, notice that while our phase $\phi$ is going to be the same as the undamped case, the amplitude is changing with time now: $R(t) = e^{\xi t}\sqrt{A^2 + B^2}$.

These cases are outlined in the following handy-dandy table:

| Type | Criterion | Solution |
|---|---|---|
| Undamped | $\gamma = 0$ | $x = A\cos\omega t + B\sin\omega t$ |
| Overdamped | $\gamma^2 - 4mk > 0$ | $x = c_1e^{r_1t} + c_2e^{r_2t}$ |
| Critically Damped | $\gamma^2 - 4mk = 0$ | $x = (c_1 + c_2t)e^{rt}$ |
| Underdamped | $\gamma^2 - 4mk < 0$ | $x = e^{\xi t(A\cos\theta t + B\sin\theta t)}$ |

The next table outlines the oscillatory behavior:

| Type | Criterion | Solution | Frequency | Period |
|---|---|---|---|---|
| Undamped | $\gamma = 0$ | $x = A\cos\omega t + B\sin\omega t$ | $\omega = \sqrt{k/m}$ | $T = 2\pi/\omega$ |
| Underdamped | $\gamma^2 - 4mk < 0$ | $x = e^{\xi t(A\cos\theta t + B\sin\theta t)}$ | $\theta = (\sqrt{4mk - \gamma^2})/2\gamma$ | $T_d = 2\pi/\theta$ |

Now lets do a couple of problems

4) We calculate the amplitude in the usual manner, $R = \sqrt{4+9} = \sqrt{13}$. And the phase, which they call $\delta$ and we call $\phi$, $\tan\delta = -2/-3$, which means we are in the third quadrant, so $\delta = \tan^{-1}(2/3) + \pi$. Finally, the frequency is $\omega = \pi$.

6) Here we must first calculate the spring constant. Recall Hooke's law, $F = kx \Rightarrow k = F/x = (.1)(9.8)/(.05) = 19.6$ N/m. Since there is no retarding force, our ODE is

$$mx'' + kx = 0;\ x(0) = 0,\ x'(0) = 0.1,$$

which has a general solution of,

$$x = A\cos\sqrt{\frac{k}{m}}t + B\sin\sqrt{\frac{k}{m}}t = A\cos 14t + B\sin 14t.$$

From the initial conditions we get, $A = 0$, $14B = 0.1 \Rightarrow B = 1/140$. This part is done incorrectly in the book because they forgot to be consistent with the units. So our solution is

$$x = \frac{1}{140}\sin 14t.$$

This means that the time of first return is $t_1 = \pi/14$, and the period is $T = \pi/7$.

Another application is the RLC circuit. Earlier in the semester we discussed the RL circuit, whcih had an ODE of $LdI/dt + RI = V$. Now, for the RLC circuit, the voltage across the conductor is $Q(t)/C = (1/C)\int_{t_0}^{t} I(\tau)d\tau + V_C(t_0)$, then our ODE becomes

$$L\frac{dI}{dt} + RI + \frac{1}{C}\int_{t_0}^{t} I(\tau)d\tau + V_C(t_0) = V. \tag{5}$$

However, we don't want an integro-differential equation, we want an ODE, so we must differentiate the entire equation to get

$$LI'' + RI' + \frac{1}{C}I = v'(t);\ I(t_0) = I_0,\ I'(t_0) = \frac{1}{L}(V(t_0) - RI_0 - Q_0/C). \tag{6}$$

However, recall $I = dQ/dt$, so we can plug this into (5) to get

$$LQ'' + RQ' + \frac{1}{C}Q = V(t);\ Q(t_0) = Q_0,\ Q'(t_0) = I(t_0) = I_0. \tag{7}$$

Notice that the equations are just like spring equations.

Lets do one electrical problem

8) For this problem we use (7). We discharge the capacitor without incoming voltage and there is no resistor, so our equation is $LQ'' + Q/C = 0$, but since $L = 1$, it is easier just to plug this in straight away, so $Q'' + Q/C = 0$. Solving for the roots gives $r^2 + 1/C = 0 \Rightarrow r = \pm i\sqrt{1/C}$. Then the general solution is

$$Q = A\cos\sqrt{1/C}t + B\sin\sqrt{1/C}t.$$

Plugging in the initial conditions gives, $Q(0) = A = 10^{-6}$, and $Q'(0) = B\sqrt{1/C} = 0 \Rightarrow B = 0$. So, our solution is

$$Q(t) = 10^{-6}\cos(2\times 10^3)t.$$

Now lets do a few more problems for oscillators in general

13) Notice that for this problem $\omega = \sqrt{k/m} = 1$. Now we solve the full ODE, which gives $r^2 + \gamma r + 1 = 0 \Rightarrow r = -\gamma/2 \pm i\sqrt{4-\gamma^2}$. So, $\theta = \sqrt{4-\gamma^2}/2$. Since we want the period to be $50 frequency to be 50\frac{1}{4}(4-\gamma^2) = 4/9 \Rightarrow 4 - \gamma^2 = 16/9 \Rightarrow \gamma = 2\sqrt{5}/3$.

19) **Overdamped:** The general solution is: $x = c_1e^{r_1t} + c_2e^{r_2t}$. Then $x = 0 \Rightarrow c_1e^{r_1t} = -c_2e^{r_2t} \Rightarrow -c_1/c_2 = e^{(r_1-r_2)t} \Rightarrow \ln(-c_1/c_2) = (r_1 - r_2)t \Rightarrow t = \ln(-c_1/c_2)/(r_2 - r_1)$, therefore $x$ has at most one zero.

**Critically Damped:** The general solution is $x = (c_1 + c_2t)e^{rt}$. Then $x = 0 \Rightarrow c_1 + c_2t = 0 \Rightarrow t = -c_1/c_2$.

Ex: Consider the IVP

$$u'' + \frac{1}{4}u' + 2u = 0;\ u(0) = 0,\ u'(0) = 2.$$

Find the time $T$ such that it is the first time the oscillation amplitude is equal to or less than half the initial oscillation amplitude.

**Solution:** The roots are $r^2 + (1/4)r + 2 = 0 \Rightarrow r = -1/8 \pm i\frac{\sqrt{127}}{8}$. Then the general solution is

$$u = e^{-t/8}\left[A\cos\frac{\sqrt{127}}{8}t + B\sin\frac{\sqrt{127}}{8}t\right]$$

So $u(0) = A = 0$ and $u'(0) = B\sqrt{127}/8 = 2 \Rightarrow B = 16/\sqrt{127}$. This gives us a solution of

$$u = \frac{16}{\sqrt{127}}e^{-t/8}\sin\frac{\sqrt{127}}{8}t.$$

The oscillation amplitude is given by $R(t) = (16/\sqrt{127})e^{-t/8}$. Then the initial oscillation amplitude is $R(0) = 16/\sqrt{127} \Rightarrow e^{-t/8} = 1$. So to get half the amplitude we need $e^{-t/8} = 1/2 \Rightarrow t/8 = \ln 2 \Rightarrow t = 8\ln 2$.

## 3.8: Applications: With Forcing

For this section we will call the natural frequency $\omega_0$.

There are many different ODEs that are possible, which we would simply solve using undetermined coefficients or variation of parameters. There are some general definitions we need to know before we can proceed.

Consider the solution

$$x = u(t) + U(t) \tag{8}$$

**Definition 1.** If $\lim_{t\to\infty}|u(t)| = 0$, $u(t)$ is called the transient solution.

**Definition 2.** If $|U(t)| < \infty$ and $\lim_{t\to\infty}|U(t)| \neq 0$, $U(t)$ is called the steady state solution; i.e. the state the system tends to as time progresses.

There is also this concept of resonance, which basically means uncontrolled vibrations due to forcing. In an undamped system if $F(t)$ contains $\sin\omega_0 t$ or equivalently $\cos\omega_0 t$, you will get a repeat, so sine and cosine terms in your $y_p$ will be multiplied by $t$ and your solution will blow up. This is resonance in an undamped system. A similar thing can happen in a damped system if the damping is small enough and the forcing frequency is very close to the natural frequency.

If there is no resonance in the system, we will have two types of oscillations in a single solution. The solution will be $y = \text{const.}(\cos\omega t - \cos\omega_0 t) = \text{const.}\left(2\sin\frac{\omega_0-\omega}{2}t\sin\frac{\omega_0-\omega}{2}t\right)$ where $\frac{1}{2}(\omega_0+\omega)$ is the high frequency oscillation and $\frac{1}{2}(\omega_0-\omega)$ is the low frequency oscillation. So there is a long time oscillation and a short time oscillation and this phenomenon is called beat.

Important identities:

$$\cos(a \pm b) = \cos a\cos b \mp \sin a\sin b \tag{9}$$

$$\sin(a \pm b) = \sin a\cos b \pm \cos a\sin b \tag{10}$$

Since there isn't enough time in the semester to delve deep into these particular concepts :(, it'll be easier to explain them with examples.

2) $\omega_0 = 7$ and $\omega = 6$, then $a = \frac{1}{2}(\omega_0+\omega)t$ and $b = \frac{1}{2}(\omega_0-\omega)t$. Then we use the sine identity to get

$$\sin(7t) = \sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$
$$\sin(6t) = \sin(a-b) = \sin(a)\cos(b) - \cos(a)\sin(b)$$

$$\Rightarrow \sin 7t - \sin 6t = 2\cos(a)\sin(b) = 2\cos\left(\frac{\omega_0+\omega}{2}t\right)\sin\left(\frac{\omega_0-\omega}{2}t\right) = 2\cos\frac{13}{2}t\sin\frac{t}{2}.$$

6) $m = 5$, $x = 0.1$, and $F = kx \Rightarrow x = \frac{5\cdot 10}{0.1} = 500$ and $\gamma = \frac{2}{0.04} = \frac{200}{4} = 50$. Then our IVP is

$$5x'' + 50x' + 500x = 10\sin\frac{t}{2};\ x(0) = 0,\ x'(0) = 0.03.$$

9) $m = 3/16$, $\gamma = 0$, $k = 12$, so the IVP is

$$\frac{3}{16}x'' + 12x = 4\cos 7t;\ x(0) = x'(0) = 0.$$

The characteristic solution will be $x_c = A_1\cos 8t + A_2\sin 8t$. Our guess for the particular solution is $y_p = B_1\cos 7t + B_2\sin 7t$, and since there are no repeats this is it. Plugging into the ODE gives us

$$-\frac{3}{16}\cdot 49B_1\cos 7t - \frac{3}{16}\cdot 49B_2\sin 7t + 12B_1\cos 7t + 12B_2\sin 7t = \frac{45}{16}B_1\cos 7t + \frac{45}{16}B_2\sin 7t$$
$$= 4\cos 7t \Rightarrow B_2 = 0, B_1 = \frac{64}{45} \Rightarrow x_p = \frac{64}{45}\cos 7t \Rightarrow x = A_1\cos 8t + A_2\sin 8t + \frac{64}{45}\cos 7t.$$

From the initial conditions $x(0) = A_1 + 64/45 = 0 \Rightarrow A_1 = -45/64$ and $x'(0) = 8A_2 = 0$. Then our full solution is

$$x = \frac{64}{45}[\cos 7t - \cos 8t] = \frac{128}{45}\sin\frac{t}{2}\sin\frac{15t}{2}.$$

12) $m = 2$, $\gamma = 1$, $k = 3$, so the ODE is

$$2x'' + x' + 3x = 3\cos 3t - 2\sin 3t.$$

For the characteristic solution, $r = \frac{1}{4}(-1 \pm \sqrt{1-24}) = -1/4 \pm i\sqrt{23}/4$, which gives us $x_c = e^{-t/4}\left[A_1\cos\frac{\sqrt{23}}{4}t + A_2\sin\frac{\sqrt{23}}{4}t\right]$. Our guess for the particular solution is $x_p = B_1\cos 3t + B_2\sin 3t$, and since there are no repeats, this is it. Plugging into the ODE gives us

$$-18B_1\cos 3t - 18B_2\sin 3t - 3B_1\sin 3t + 3B_2\cos 3t + 3B_1\cos 3t + 3B_2\sin 3t$$
$$= (3B_2 - 15B_1)\cos 3t - (3B_1 + 15B_2)\sin 3t = 3\cos 3t - 2\sin 3t.$$

This gives us two equations with two unknowns: $3B_2 - 15B_1 = 3$ and $3B_1 + 15B_2 = 2$, then $B_1 = -1/6$ and $B_2 = 1/6$. So we get

$$x_p = -\frac{1}{6}\cos 3t + \frac{1}{6}\sin 3t \Rightarrow x = e^{-t/4}\left[A_1\cos\frac{\sqrt{23}}{4}t + A_2\sin\frac{\sqrt{23}}{4}t\right] + \frac{1}{6}[\sin 3t - \cos 3t].$$

Notice the terms with the exponent die off as $t$ goes to infinity, however, the other terms remain, so the steady state solution is

$$x_\infty = \frac{1}{6}[\sin 3t - \cos 3t] = \frac{\sqrt{2}}{6}\cos\left(3t - \frac{3\pi}{4}\right)$$

18) (a) The characteristic solution is easily found to be $u_c = A_1\cos t + A_2\sin t$. Our guess for the particular solution is $u_p = B_1\cos\omega t + B_2\sin\omega t$, and since $\omega \neq 1$, there are no repeats. Plugging into the ODE gives us

$$-\omega^2B_1\cos\omega t - \omega^2B_2\sin\omega t + B_1\cos\omega t + B_2\sin\omega t = (1-\omega^2)B_1\cos\omega t + (1-\omega^2)B_2\sin\omega t$$
$$= 3\cos\omega t \Rightarrow B_2 = 0, B_1 = \frac{3}{1-\omega^2} \Rightarrow u_p = \frac{3}{1-\omega^2}\cos\omega t \Rightarrow u = A_1\cos t + A_2\sin t + \frac{3}{1-\omega^2}\cos\omega t.$$

The initial conditions gives us $u(0) = A_1 + \frac{3}{1-\omega^2} = 0 \Rightarrow A_1 = -3/(1-\omega^2)$ and $u'(0) = A_2 = 0$. Then our solution is

$$u = \frac{3}{1-\omega^2}[\cos\omega t - \cos t] = \frac{6}{1-\omega^2}\sin\frac{1-\omega}{2}t\sin\frac{1+\omega}{2}t.$$

If we did have $\omega = 1$ in the beginning, then notice that there would be a repeat and our solution would have a multiple of t, which would blow up.

(b) As $\omega \to 1$, the amplitude increases: $|u_{\max}| \to \infty$.