(1) Find α such that $y_1 = x^{1/2}$ is a solution to the ODE

$$x^2y'' + \alpha xy' + y = 0$$

and find the other linearly independent solution y_2 (hint: it's easier than it looks).

Solution: The characteristic polynomial is

$$r(r-1) + \alpha r + 1 = r^2 + (\alpha - 1)r + 1 = 0.$$

Now, we know one of the roots is $r = 1/2$, and we notice that the last term of the polynomial is 1, so the other root must be $r = 2$, then the solution is $y_2 = x^2$ and our characteristic polynomial is

$$r^2 - \frac{5}{2}r + 1 = 0 \Rightarrow \alpha = -\frac{3}{2}.$$

(2) Find all singular points and determine their regularity for the following ODE

$$(1 - x^2)y'' - 2xy' + \beta(\beta + 1)y = 0.$$

Solution: We first put this into standard form,

$$y'' - \frac{2x}{1 - x^2}y' + \frac{\beta(\beta + 1)}{1 - x^2}y = 0; \ x_0 = \pm 1.$$

Then we use the following limits,

$$x_0 = 1: \quad \lim_{x \to 1} \cancel{(x-1)} \cdot \frac{2x}{\cancel{(x-1)}(x+1)} = 1\checkmark \quad \lim_{x \to 1} (x-1)^{\cancel{2}} \cdot \frac{\beta(\beta+1)}{\cancel{(1-x)}(1+x)} = 0\checkmark$$

$$x_0 = -1: \quad \lim_{x \to -1} \cancel{(x+1)} \cdot \frac{2x}{\cancel{(x+1)}(x-1)} = 1\checkmark \quad \lim_{x \to -1} (x+1)^{\cancel{2}} \cdot \frac{\beta(\beta+1)}{\cancel{(1+x)}(1-x)} = 0\checkmark$$

All the limits are convergent, so both points are regular singular points.

(3) Consider the power series solution to the ODE $y'' + y = 0$.

Solution: First plug in the series solution Ansatz to get,

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + a_nx^n = 0.$$

(a) Find the recurrence relation.

Solution: Since both exponents are n we can go straight to the general term,

$$a_{n+2} = -\frac{a_n}{(n+2)(n+1)}.$$

(b) Write out the first two nonzero terms for the two linearly independent solutions.

Solution: For the two solutions we have

$$a_0 = 0, \ a_1 = 1 \Rightarrow a_2 = a_4 = \cdots = 0; \ a_3 = -\frac{1}{6} \Rightarrow y_1 = x - \frac{1}{6}x^3 + \cdots$$

$$a_1 = 0, \ a_0 = 1 \Rightarrow a_3 = a_5 = \cdots = 0 \ \ a_2 = -\frac{1}{2} \Rightarrow y_2 = 1 - \frac{1}{2}x^2 + \cdots$$

(c) Determine the radius of convergence for each series by using the ratio test.

Solution: Recall the ratio test,

$$\lim_{n \to \infty} \left| \frac{x^{n+2}a_{n+2}}{x^na_n} \right| = \lim_{n \to \infty} x^2 \frac{1}{(n+2)(n+1)} = 0 < 1$$

Since this is true for all x the radius of convergence is $R = \infty$.

(4) Consider the function

$$g(t) = \begin{cases} e^{-t} & 0 \le t < 1 \\ e^{-3t} + 1 & 1 \le t < 2 \\ 1 & t \ge 2 \end{cases}$$

(a) Graph the function for $0 \le t \le 3$.
Solution: Graph it!

(b) Write $g(t)$ as unit step functions; i.e. the "u" notation.
Solution: We need to break $g(t)$ up into a few two-step step-functions.

$$g(t) = e^{-t} + \begin{cases} 0 & t < 1, \\ e^{-3t} + 1 - e^{-t} & 1 \le t < 2, \\ 1 - e^{-t} & t \ge 2; \end{cases} = e^{-t} + \begin{cases} 0 & t < 1, \\ e^{-3t} + 1 - e^{-t} & t \ge 1; \end{cases} - \begin{cases} 0 & t < 2, \\ e^{-3t} & t \ge 2; \end{cases}$$

$$= e^{-t} + e^{-3t}u_1(t) + u_1(t) - e^{-t}u_1(t) - e^{-3t}u_2(t)$$

$$= e^{-t} + e^{-3}e^{-3(t-1)}u_1(t) + u_1(t) - e^{-1}e^{-(t-1)}u_1(t) - e^{-6}e^{-3(t-2)}u_2(t).$$

(c) Find the Laplace transform of the function.
Solution: Taking the Laplace of each term gives us,

$$\frac{1}{s+1} + e^{-3}\frac{1}{s+3}e^{-s} + \frac{1}{s}e^{-s} - e^{-1}\frac{1}{s+1}e^{-s} - e^{-6}\frac{1}{s+3}e^{-2s}.$$

(5) Solve the following IVP

$$y'' + 4y' + 8y = 2u_\pi(t) - 2\delta(t - 2\pi);\ y(0) = 2,\ y'(0) = 0$$

Solution: Taking the Laplace Transform of the IVP give us

$$-\cancelto{0}{y'(0)} - s\cancelto{2}{y(0)} + s^2Y - 4\cancelto{2}{y(0)} + 4sY + 8Y = 2e^{-\pi s}\frac{1}{s} - 2e^{-2\pi s}$$

$$\Rightarrow (s^2 + 4s + 8)Y = 2e^{-\pi s}\frac{1}{s} - 2e^{-2\pi s} + 2s + 8$$

$$\Rightarrow Y = 2\frac{s+4}{(s+2)^2 + 4} - 2e^{-2\pi s}\frac{1}{(s+2)^2+4} + 2e^{-\pi s}\frac{1}{s(s^2+4s+8)}.$$

The last term needs to be separated using partial fractions,

$$\frac{1}{s(s^2+4s+8)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 4s + 8} \Rightarrow As^2 + 4As + 8A + Bs^2 + Cs = (A+B)s^2 + (4A + C)s + 8A = 1$$

So, $A = \frac{1}{8} \Rightarrow C = -\frac{1}{2},\ B = -\frac{1}{8}$. Then we get

$$Y = 2 \cdot \frac{s+2}{(s+2)^2+4} + 2\frac{2}{(s+2)^2+4} - e^{-2\pi s}\frac{2}{(s+2)^2+4} + e^{-\pi s}\left[\frac{1/4}{s} - \frac{1}{4} \cdot \frac{s+2}{(s+2)^2+4} - \frac{1}{4}\frac{2}{(s+2)^2+4}\right]$$

$$\Rightarrow y = 2e^{-2t}\cos 2t + 2e^{-2t}\sin 2t - u_{2\pi}(t)e^{-2(t-2\pi)}\sin 2(t - 2\pi)$$

$$+ u_\pi(t)\left[\frac{1}{4} - \frac{1}{4}e^{-2(t-\pi)}\cos 2(t-\pi) - \frac{1}{4}e^{-2(t-\pi)}\sin 2(t-\pi)\right].$$

(6) Use a convolution to find the Laplace Transform of (Don't integrate)

$$F(s) = \frac{1}{s^3(s^2+1)}.$$

Solution: The inverse Laplace for $1/s^3$ is $1/2t^2$ and for $1/(s^2+1)$ it's $\sin t$, so we get

$$f(t) = \int_0^t \frac{1}{2}(t-\tau)^2 \sin\tau d\tau$$

(7) Find the Laplace Transform of

$$f(t) = \int_0^t (t-\tau)^2 \cos(2\tau) d\tau$$

Solution: As before, the Laplace of t^2 is $2/s^3$ and for $\cos 2t$ it is $s/(s^2+4)$, so we get

$$F(s) = \frac{2s}{s^3(s^2+4)}$$

(8) Find the inverse Laplace Transform (in closed form) of

$$F(s) = \frac{s^2-9}{s^3+6s^2+9s}.$$

Solution: We first simplify F

$$F(s) = \frac{\cancel{(s+3)}(s-3)}{s(s+3)^{\cancel{2}}} = \frac{1}{s+3} - \frac{3}{s(s+3)} = \frac{1}{s+3} - \frac{1}{s} + \frac{1}{s+3} = \frac{2}{s+3} - \frac{1}{s}.$$

Then the inverse Laplace Transform is $f(t) = 2e^{-3t} - 1$.

(9) Find the inverse Laplace Transform (in closed form) of

$$G(s) = e^{-s}\frac{s-2}{s^2+2s+2}$$

Solution: We first break it up and then take the inverse Laplace Transform,

$$G(s) = e^{-s}\left[\frac{s+1}{(s+1)^2+1} - 3\frac{1}{(s+1)^2+1}\right] \Rightarrow g(t) = u_1(t)\left[e^{-(t-1)}\cos(t-1) - 3e^{-(t-1)}\sin(t-1)\right].$$

(10) Find the inverse Laplace Transform of

$$F(s) = \frac{3}{s^2+4}$$

Solution: For this we get,

$$F(s) = \frac{3}{2}\cdot\frac{2}{s^2+4} \Rightarrow f(t) = \frac{3}{2}\sin 2t.$$

(11) Find the inverse Laplace Transform of

$$F(s) = \frac{2s-3}{s^2-4}$$

solution: Again, we split it up and take the inverse Laplace,

$$F(s) = 2\frac{s}{s^2-4} - \frac{3}{2}\frac{2}{s^2-4} \Rightarrow f(t) = 2\cosh 2t - \frac{3}{2}\sinh 2t.$$

(12) Find the inverse Laplace Transform of

$$F(s) = \frac{1-2s}{s^2+2s+10}$$

Solution: And once more,

$$F(s) = -2\frac{s+1}{(s+1)^2+9} + \frac{3}{(s+1)^2+9} \Rightarrow f(t) = -2e^{-t}\cos 3t + e^{-t}\sin 3t.$$

(13) Use Laplace Transforms to solve the IVP

$$y^{(4)} - y = 0;\ y(0) = 1,\ y'(0) = 0,\ y''(0) = -2,\ y'''(0) = 0.$$

Solution: We take the Laplace of the IVP, solve for Y, and take the inverse transform,

$$-\cancelto{0}{y'''(0)} - s\cancelto{-2}{y''(0)} - s^2\cancelto{0}{y'(0)} - s^3\cancelto{1}{y(0)} + s^4Y - Y = 0 \Rightarrow (s^4-1)Y = s^3 - 2s$$
$$\Rightarrow Y = \frac{s(s^2-1)-s}{s^4-1} = \frac{s}{s^2+1} - \frac{s}{s^4-1} = \frac{s}{s^2+1} - \frac{s}{2}\left[\frac{1}{s^2-1} - \frac{1}{s^2+1}\right] = \frac{3}{2}\frac{s}{s^2+1} - \frac{1}{2}\frac{s}{s^2-1}$$
$$\Rightarrow y = \frac{3}{2}\cos t - \frac{1}{2}\cosh t$$

(14) Use Laplace Transforms to solve the IVP

$$y'' + 4y' = \begin{cases} t & 0 \le t < 1, \\ 0 & t \ge 1 \end{cases}; \ y(0) = y'(0) = 0$$

Solution: We first put it into a form that we can take the Laplace of

$$y'' + 4y' = t - \begin{cases} 0 & 0 \le t < 1, \\ t & t \ge 1 \end{cases} = t - tu_1(t) = t - (t-1)u_1(t) - u_1(t).$$

Then we take the Laplace of the IVP

$$(s^2+4s)Y = \frac{1}{s^2} - e^{-s}\frac{1}{s^2} - e^{-s}\frac{1}{s} \Rightarrow Y = \frac{1}{s^2(s^2+4s)} - e^{-s}\frac{s+1}{s^2(s^2+4s)}.$$

Then we take the partial fractions, however we have to do two. Yea yea, I know it's a pain, but I'm not the one who made the problem.

$$\frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s+4} \Rightarrow As^3 + 4As^2 + Bs^2 + 4Bs + Cs + 4C + Ds^3 = (A+D)s^3 + (4A+B)s^2 + (4B+C)s + 4C = 1.$$

Then the constants are $C = 1/4$, $B = -1/16$, $A = 1/64$, $D = -1/64$. For the second partial fractions we have

$$(A+D)s^3 + (4A+B)s^2 + (4B+C)s + 4C = s+1$$

Then the constants are $C = 1/4$, $B = 3/16$, $A = -3/64$, $D = 3/64$. Putting these back into the equation gives us

$$Y = \frac{1/64}{s} - \frac{1/16}{s^2} + \frac{1/4}{s^3} - \frac{1/64}{s+4} - e^{-s}\left[-\frac{3/64}{s} + \frac{3/16}{s^2} + \frac{1/4}{s^3} + \frac{3/64}{s+4}\right]$$
$$\Rightarrow y = \frac{1}{64} - \frac{1}{16}t + \frac{1}{8}t^2 - \frac{1}{64}e^{-4t} - u_1(t)\left[-\frac{3}{64} + \frac{3}{16}(t-1) + \frac{1}{8}(t-1)^2 + \frac{3}{64}e^{-4(t-1)}\right].$$

(15) Use Laplace Transforms to solve the IVP

$$y' + ay = e^{\lambda t}; \ y(0) = c,$$

with $a \ne 0$. What happens to the solution when $\lambda + a \ne 0$? What about for $\lambda + a = 0$?

Solution: First lets take the Laplace Transform of the IVP

$$-\cancelto{c}{y(0)} + sY + aY = \frac{1}{s-\lambda} \Rightarrow (s+a)Y = c + \frac{1}{s-\lambda} \Rightarrow Y = \frac{c}{s+a} + \frac{1}{(s+a)(s-\lambda)}$$

Lets assume that $\lambda + a \ne 0$, then we proceed as usual

$$Y = \frac{c}{s+a} + \frac{-1}{a+\lambda}\left[\frac{1}{s+a} - \frac{1}{s-\lambda}\right] = \left(c - \frac{1}{a+\lambda}\right)\frac{1}{s+a} + \frac{1}{a+\lambda}\cdot\frac{1}{s-\lambda}$$
$$y = \left(c - \frac{1}{a+\lambda}\right)e^{-at} + \frac{1}{a+\lambda}e^{\lambda t} \qquad \text{if } \lambda + a \ne 0$$

Now, if $\lambda + a = 0$, $\lambda = -a$, then

$$Y = \frac{c}{s+a} + \frac{1}{(s+a)^2} \Rightarrow y = ce^{-at} + te^{-at}.$$