5.1 Power Series Review

Know it!

5.2 Series Solutions

Consider the ODE,

$$y^{(n)} + F_{n-1}(x)y^{(n-1)} + \cdots + F_1(x)y' + F_0(x)y = Q(x). \tag{1}$$

First lets define a few things,

Definition 1. A point $x = x_0$ is said to be an ordinary point of (1) if $F_n, \ldots, F_0, Q$, all have convergent Taylor series in a neighborhood of x_0. However if at least one function does not satisfy this criterion, $x = x_0$ is called a singular point.

For this section we consider the problem,

$$P(x)y'' + Q(x)y' + R(x)y = 0;\ x = x_0, \tag{2}$$

where $x = x_0$ is an ordinary point and R, Q, P are polynomials. We make the ansatz:

$$y = \sum_{n=0}^{\infty} a_n(x-x_0)^n \tag{3}$$

$$\Rightarrow y' = \sum_{n=1}^{\infty} na_n(x-x_0)^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}(x-x_0)^n \tag{4}$$

$$\Rightarrow y'' = \sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-x_0)^n. \tag{5}$$

Then we plug this into the ODE (2) and try to solve for the "a's".

Lets do some problems,

3) (a) Plugging into the ODE gives,

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n - (n+1)a_{n+1}x(x-1)^n - a_n(x-1)^n = 0,$$

but $x = 1 + (x-1)$, so

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n - (n+1)a_{n+1}(x-1)^n - (n+1)a_{n+1}(x-1)^{n+1} - a_n(x-1)^n = 0$$

By matching terms we get:

$$x^0: \qquad 2a_2 - a_1 - a_0 = 0 \Rightarrow a_2 = \frac{1}{2}(a_1 + a_0),$$

$$x^m: \qquad (m+2)(m+1)a_{m+2} - (m+1)a_{m+1} - (m+1)a_m = 0 \Rightarrow a_{m+2} = \frac{a_{m+1} + a_m}{m+2} \text{ for } m \geq 1$$

Notice that we can not solve for a_1 and a_0 because these are like our c_1 and c_0 where we have to solve for them using the initial conditions, if given.

(b) This means $a_0 = 0$ gives one solution and $a_1 = 0$ gives another,

$$a_0 = 0 \Rightarrow a_2 = \frac{a_1}{2} \Rightarrow a_3 = \frac{a_1}{2} \Rightarrow a_4 = \frac{a_1}{4} \cdots \Rightarrow y_2 = (x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{4}(x-1)^3 + \frac{1}{4}(x-1)^4 + \cdots$$

$$a_1 = 0 \Rightarrow a_2 = \frac{a_0}{2} \Rightarrow a_3 = \frac{a_0}{6} \Rightarrow a_4 = \frac{a_0}{6} \cdots \Rightarrow y_1 = 1 + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{6}(x-1)^4 + \cdots$$

6) (a) Plugging in to the ODE gives,

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(2+x^2)x^n - (n+1)a_{n+1}x^{n+1} + 4a_n x^n$$

$$= \sum_{n=0}^{\infty} 2(n+2)(n+1)a_{n+2}x^n + (n+2)(n+1)a_{n+2}x^{n+2} - (n+1)a_{n+1}x^{n+1} + 4a_n x^n = 0$$

By matching terms we get:

x^0 : $\quad 4a_2 + 4a_0 = 0 \Rightarrow a_2 = -a_0$

x^1 : $\quad 12a_3 + 3a_1 = 0 \Rightarrow a_3 = -\frac{1}{4}a_1$

x^m : $\quad 2(m+2)(m+1)a_{m+2} + m(m-1)a_m - ma_m + 4a_m = 0 \Rightarrow a_{m+2} = -\frac{m^2-2m+4}{2(m+2)(m+1)}a_m;\ m \geq 2$

(b) Now we find the first few terms of our two solutions,

$$a_0 = 0 \Rightarrow a_2 = a_4 = \cdots = 0,\ \text{ so } a_3 = -\frac{1}{4}a_1 \Rightarrow a_5 = \frac{7}{160}a_1 \Rightarrow y_2 = x - \frac{1}{4}x^3 + \frac{7}{160}x^5 + \cdots$$

$$a_1 = 0 \Rightarrow a_3 = a_5 = \cdots = 0,\ \text{ so } a_2 = -a_0 \Rightarrow a_4 = \frac{1}{6}a_0 \Rightarrow y_1 = 1 - x^2 + \frac{1}{6}x^3 + \cdots.$$

11) (a) Plugging into the ODE gives,

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(3-x^2)x^n - 3(n+1)a_{n+1}x^{n+1} - a_n x^n$$

$$= \sum_{n=0}^{\infty} 3(n+2)(n+1)a_{n+2}x^n - (n+2)(n+1)a_{n+2}x^{n+2} - 3(n+1)a_{n+1}x^{n+1} - a_n x^n = 0$$

Matching terms gives,

x^0 : $\quad 6a_2 - a_0 = 0 \Rightarrow a_2 = \frac{1}{6}a_0$

x^1 : $\quad 18a_3 - 3a_1 - a_1 = 18a_3 - 4a_1 = 0 \Rightarrow a_3 = \frac{2}{9}a_1$

x^m : $\quad 3(m+2)(m+1)a_{m+2} - m(m-1)a_m - 3ma_m - a_m = 0$

$$\Rightarrow a_{m+2} = \frac{1+3m+m^2-m}{3(m+2)(m+1)}a_m = \frac{m+1}{3(m+2)}a_m;\ m \geq 2$$

(b) For the first few terms we get,

$$a_0 = 0 \Rightarrow a_2 = a_4 = \cdots = 0 \Rightarrow a_3 = \frac{2}{9}a_1 \Rightarrow a_5 = \frac{4}{15}a_3 = \frac{8}{135}a_1 \Rightarrow y_2 = x + \frac{2}{9}x^3 + \frac{8}{135}x^5 + \cdots$$

$$a_1 = 0 \Rightarrow a_3 = a_5 = \cdots = 0 \Rightarrow a_2 = \frac{1}{6}a_0 \Rightarrow a_4 = \frac{3}{12}a_2 = \frac{1}{24}a_0 \Rightarrow y_1 = 1 + \frac{1}{6}x^2 + \frac{1}{24}x^4 + \cdots.$$

5.4 Euler's Equation; Regular singular points

Consider the ODE,

$$x^2y''(x) + \alpha xy'(x) + \beta y(x) = 0. \tag{6}$$

This has a singular point because if we put this into standard form we get,

$$y'' + \alpha\frac{1}{x}y' + \beta\frac{1}{x^2}y = 0,$$

which violates the existence and uniqueness theorem at $x = 0$. We obviously don't know how to deal with this problem. But there is a similar problem we know how to deal with really well,

$$y''(\xi) + ay'(\xi) + by(\xi) = 0. \tag{7}$$

Basically we need to make a change of variables on x in order to get rid of the x's in the coefficients. What do we know that gives us $1/x$ every time we differentiate? $\xi = \ln x$ does the trick. Taking the derivatives are a little different than what we are used to, but very intuitive due to Leibniz notation,

$$\frac{dy}{dx} = \frac{dy}{d\xi}\frac{d\xi}{dx} = \frac{1}{x}\frac{dy}{d\xi},$$
$$\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{dy'}{d\xi}\frac{d\xi}{dx} = \frac{1}{x}\left(e^{-\xi}\frac{dy}{d\xi}\right)' = \frac{1}{x}\left(-e^{-\xi}\frac{dy}{d\xi} + e^{-\xi}\frac{d^2y}{dx^2}\right) = \frac{1}{x^2}\left(\frac{d^2y}{d\xi^2} - \frac{dy}{d\xi}\right).$$

Plugging this back into (7) gives us,

$$\frac{d^2y}{d\xi^2} - \frac{dy}{d\xi} + \alpha\frac{dy}{d\xi} + \beta y = y'' + ay' + by = 0.$$

To solve (7) we used the ansatz $y = \exp(r\xi)$, so to solve (6) we use $y = x^r$. Lets think of a slightly more general second order ODE for this part,

$$Ax^2y'' + Bxy' + Cy = 0.$$

Then plugging into this gives,

$$Ax^2[r(r-1)]x^{r-2} + Bxrx^{r-1} + Cx^r = Ar(r-1)x^r + Brx^r + Cx^r = 0 \Rightarrow Ar(r-1) + Br + C = 0.$$

This is our characteristic polynomial of Euler's equation. And we have the usual cases,

Cases	Solution	Comment
Distinct Roots	$y = c_1x^{r_1} + c_2x^{r_2}$	
Repeated Roots	$y = (c_1 + c_2\ln x)x^r$	because $\xi = \ln x$
Complex Conjugate Roots	$y = x^\lambda(A\cos(\mu\ln x) + B\sin(\mu\ln x))$	where $r = \lambda \pm i\mu$

Now lets do some problems,

5) The characteristic polynomial is $r(r-1)-r+1 = r^2-2r+1 = (r-1)^2 = 0$, so we have repeated roots $r = 1$, then $y = (c_1 + c_2 \ln|x|)x$; $x \neq 0$.
12) The characteristic polynomial is $r(r-1) - 4r + 4 = r^2 - 5r + 4 = (r-1)(r-4) = 0$, then $y = c_1x + c_2x^4$; $x \neq 0$.
11) The characteristic polynomial is $r(r-1) + 2r + 4 = r^2 + r + 4 = 0$, then $r = (-1 \pm i\sqrt{15})/2$, so

$$y = |x|^{-1/2}\left[A\cos\left(\frac{\sqrt{15}}{2}\ln|x|\right) + B\sin\left(\frac{\sqrt{15}}{2}\ln|x|\right)\right].$$

There is one more small theoretical thing we have to discuss,

Definition 2. Suppose $x = x_0$ is a singular point of the ODE

$$y'' + P(x)y' + Q(x)y = 0.$$

If $(x-x_0)P(x)$ and $(x-x_0)^2Q(x)$ have convergent Taylor series at $x = x_0$, then $x = x_0$ is called a <u>regular singular point</u>. Otherwise it is called an <u>irregular singular point</u>.

Lets do one example of this,

19) We convert this to standard form,

$$y'' + \frac{x-2}{x^2(1-x)}y' - \frac{3}{x(1-x)}y = 0$$

So, our singular points are $x = 0, 1$. Since these are polynomials it suffices to take the limit and see if it converges,

$$x = 0: \lim_{x\to 0} xP(x) = \lim_{x\to 0} x\frac{x-2}{x^2(1-x)} = \frac{x-2}{x(1-x)} = \infty.$$

So, $x = 0$ is an irregular singular point.

$$\lim_{x\to 1}(x-1)P(x) = \lim_{x\to 1}(x-1)\frac{x-2}{x^2(1-x)} = \lim_{x\to 1}\frac{2-x}{x^2} = 1.$$
$$\lim_{x\to 1}(x-1)^2Q(x) = \lim_{x\to 1}(x-1)^2\frac{-3}{x(1-x)} = \lim_{x\to 1}\frac{3(1-x)}{x} = 0.$$

This means that $x = 1$ is a regular singular point.

6.1 Laplace Transforms

Differential equations are hard! With the characteristic polynomial we were able to convert the problem into an algebraic equation, but this only works for simple problems. For harder problems there are special types of transforms called integral transforms,

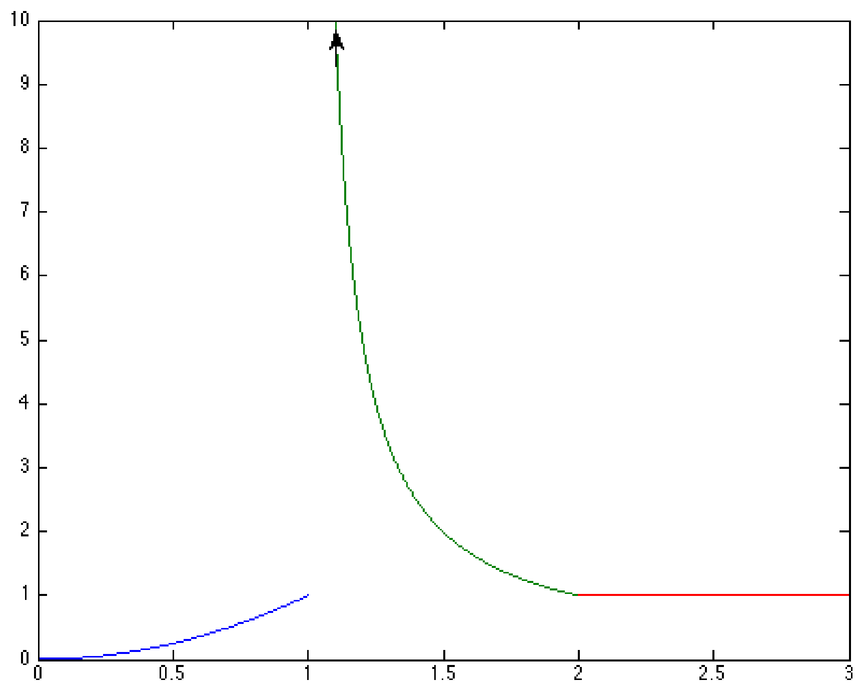
Definition 3. If $f(t)$ is defined for all $t > 0$ and if $s \in \mathbb{R}$ such that the integral

$$F(s) = \int_0^\infty e^{-st} f(t) dt \tag{8}$$

converges for $s < s_n < \infty$, then $F(s)$ is called the Laplace Transform of $f(t)$ and denoted as $\mathcal{L}\{f(t)\} = F(s)$.

Now lets do some examples,

2) The sketch should look like,



10) Here we use the definition of the Laplace Transform,

$$F(s) = \int_0^\infty e^{-st}e^{at}\sinh bt = \int_0^\infty e^{-st}e^{at}\frac{1}{2}(e^{bt}-e^{-bt})dt = \frac{1}{2}\int_0^\infty e^{(b+a-s)t} - e^{(-b+a-s)t}dt$$

$$= \lim_{\tau\to\infty}\frac{1}{2}\int_0^\tau e^{(b+a-s)t} - e^{(-b+a-s)t}dt = \frac{1}{2}\lim_{\tau\to\infty}\left[\frac{1}{b+a-s}e^{(b+a-s)t} - \frac{1}{-b+a-s}e^{(-b+a-s)t}\right]_0^\tau$$

$$= \frac{1}{2}\lim_{\tau\to\infty}\left[\frac{1}{b+a-s}e^{(b+a-s)\tau} - \frac{1}{-b+a-s}e^{(-b+a-s)\tau} - \frac{1}{b+a-s} + \frac{1}{-b+a-s}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a-b} - \frac{1}{s-a+b}\right] = \frac{b}{(s-a)^2-b^2}.$$

However, notice that we can only take the integral if both $b+a-s<0$ and $-b+a-s<0$, which means we need $s-a>|b|$.

14) Again we use the definition,

$$F(s) = \int_0^\infty e^{-st}e^{at}\cos bt = \int_0^\infty e^{-st}e^{at}\frac{1}{2}(e^{ibt}-e^{-ibt})dt = \frac{1}{2}\int_0^\infty \left(e^{[(a+ib)-s]t} - e^{[(a-ib)-s]t}\right)dt$$

$$= \lim_{\tau\to\infty}\frac{1}{2}\int_0^\tau \left(e^{[(a+ib)-s]t} - e^{[(a-ib)-s]t}\right)dt = \frac{1}{2}\lim_{\tau\to\infty}\left[\frac{1}{(a+ib)-s}e^{[(a+ib)-s]t} - \frac{1}{(a-ib)-s}e^{[(a-ib)-s]t}\right]_0^\tau$$

$$= \frac{1}{2}\lim_{\tau\to\infty}\left[\frac{1}{(a+ib)-s}e^{[(a+ib)-s]\tau} - \frac{1}{(a-ib)-s}e^{[(a-ib)-s]\tau} - \frac{1}{(a+ib)-s} + \frac{1}{(a-ib)-s}\right]$$

$$= \frac{1}{2}\left[\frac{1}{(a-ib)-s} - \frac{1}{(a+ib)-s}\right] = \frac{s-a}{(s-a)^2+b^2}.$$

The condition for this is more difficult, but if we ignore the complex part, which we can do because of certain properties of complex numbers, we get that the integral converges for $s>a$.

22) This one is much easier than the previous two. Notice that after $t=1$ the function is zero, so our integral becomes,

$$F(s) = \int_0^\infty e^{-st}f(t)dt = \int_0^1 te^{-st}dt = -\frac{1}{s^2}e^{-st}(st+1)\Big|_0^1 = -\frac{1}{s^2}e^{-s}(s+1) + \frac{1}{s^2}.$$

We have one more small theoretical consideration. This can be derived using Calc II, but unless you were in my class you probably didn't see this in Calc II.

Theorem 1. *If $|f(t)| \le g(t)$ for $t \ge M$, $\int_M^\infty g(t)dt$ converges implies $\int_a^\infty f(t)dt$ also converges for $t \ge a$, and*

If $f(t) \ge g(t)$ for $t \ge M$, $\int_M^\infty g(t)dt$ diverges implies $\int_0^\infty f(t)dt$ diverges for $t \ge a$.

Lets do one example with this,

28) We have that $|\cos t| \le 1$, so $|e^{-t}\cos t| \le e^{-t}$, and $\int_0^\infty e^{-t}dt = 1$ converges, so $\int_0^\infty e^{-t}\cos t dt$ also converges.