6.5 Impulse Functions

An impulse is a change of momentum over a period of time. This can range from hitting a baseball to a punch to the face. The following plot gives an illustration of this,

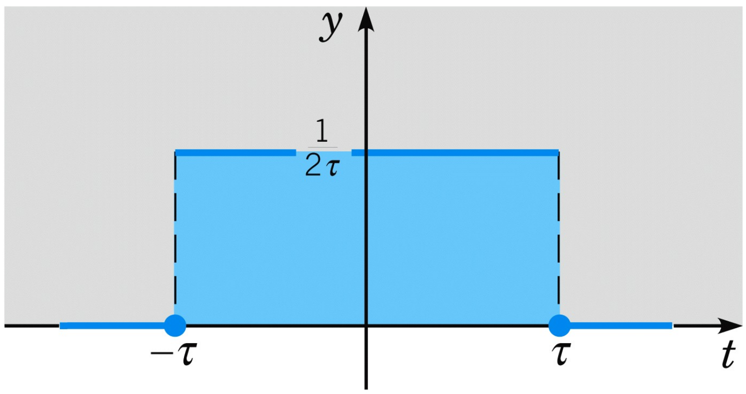


Figure 6.5.1

The momentum here is,

$$p = \int_{-\infty}^{\infty} g(t)dt = \int_{-\tau}^{\tau} 1/(2\tau)dt = 1.$$

Notice that we can make τ smaller and keep the momentum at $p = 1$ such as in the following plot,

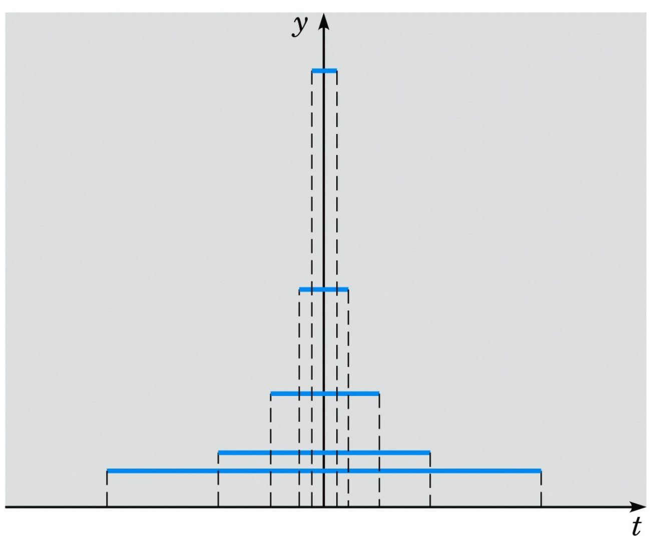


Figure 6.5.2

In fact,

$$\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} g(t)dt = 1.$$

Notice this is 0 everywhere except at $t = 0$. Now, if we can do this at $t = 0$ we can define a "function" with this property for any $t = t_0$,

$$\int_{-\infty}^{\infty} \delta(t-t_0)dt = 1;\ \delta(t-t_0) = 0 \quad \forall t \neq t_0 \tag{1}$$

called the Dirac delta function, however this isn't a function, but rather a distribution. Doing this for $t_0 > 0$ will allow us to employ laplace transforms. Notice that we can write the delta function as the following limit,

$$\delta(t-t_0) = \lim_{\epsilon\to 0}\begin{cases} 0 & t \leq t_0-\epsilon, \\ \frac{1}{2\epsilon} & t_0-\epsilon < t < t_0+\epsilon, \\ 0 & t \geq t_0+\epsilon; \end{cases} \quad = \lim_{\epsilon\to 0}\frac{1}{2\epsilon}\left(u_{t_0-\epsilon}(t) - u_{t_0+\epsilon}(t)\right).$$

Now we take the laplace transform,

$$\mathcal{L}\{\delta(t-t_0)\} = \lim_{\epsilon\to 0}\frac{1}{2\epsilon}\cdot\frac{1}{s}\left(e^{(-t_0+\epsilon)s} - e^{(-t_0-\epsilon)s}\right) = e^{-t_0 s}\lim_{\epsilon\to 0}\frac{1}{\epsilon s}\cdot\frac{1}{2}\left(e^{\epsilon s} - e^{-\epsilon s}\right)$$

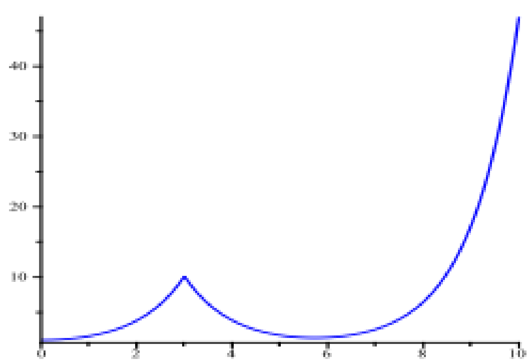
$$= e^{-t_0 s}\cancelto{1}{\lim_{\epsilon\to 0}\frac{\sinh \epsilon s}{\epsilon s}} = e^{-t_0 s}. \tag{2}$$

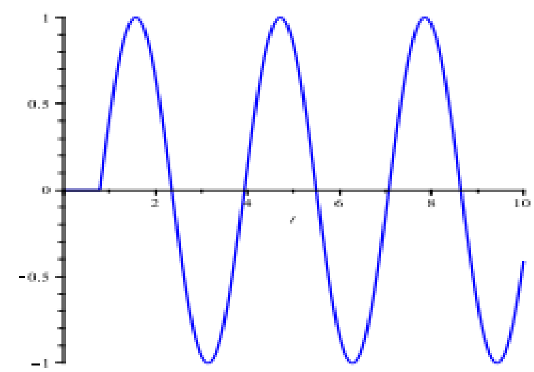
Now, lets do some problems,

4) We take the laplace transform of the entire ODE (Plot on left),

$$-\cancelto{0}{y'(0)} - s\cancelto{1}{y(0)} + s^2Y - Y = -20e^{-3s} \Rightarrow (s^2-1)Y = -20e^{-3s} \Rightarrow Y = \frac{1}{s^2-1}\left(-20e^{-3s}+s\right)$$

$$\Rightarrow y = \cosh t - 20\sinh(t-3)u_3(t).$$





8) We take the laplace transform of the entire ODE (Plot on right),

$$-\cancelto{0}{y'(0)} - s\cancelto{0}{y(0)} + s^2Y + 4Y = 2e^{-(\pi/4)s} \Rightarrow Y = \frac{2}{s^2+4}e^{-(\pi/4)s}$$

$$\Rightarrow y = \sin(2(t-\pi/4))u_{\pi/4}(t) = (\cos 2t)u_{\pi/4}(t).$$

11) As per usual,

$$(s^2+2s+2)Y = \frac{s}{s^2+1} + e^{-(\pi/2)s} \Rightarrow Y = \frac{s}{(s^2+1)(s^2+2s+2)} + \frac{e^{-(\pi/2)s}}{s^2+2s+2}.$$

We employ partial fractions,

$$\frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2s+2} = \frac{s}{(s^2+1)(s^2+2s+2)} \Rightarrow As^3+2As^2+2As+Bs^2+2Bs+2B+Cs^3+Cs+Ds^2+D = s$$
$$\Rightarrow (A+C)s^3+(2A+B+D)s^2+(2A+2B+C)s+(2B+D) = s.$$

From this we get $A = 1/5 = -C$, $B = 2/5$, and $D = -4/5$, so

$$Y = \frac{1}{5}\left[\frac{s}{s^2+1} + \frac{2}{s^2+1} - \frac{s+4}{s^2+2s+2}\right] + e^{-(\pi/2)s}\frac{1}{(s+1)^2+1}.$$

Furthermore,

$$\frac{s+4}{(s+1)^2+1} = \frac{s+1}{(s+1)^2+1} + \frac{3}{(s+1)^2+1}.$$

Then,

$$y = \frac{1}{5}\cos t + \frac{2}{5}\sin t - \frac{1}{5}e^{-t}(\cos t + 3\sin t) + e^{-(t-\pi/2)}\sin(t-\pi/2)u_{pi/2}(t).$$

6.6 Convolutions

To derive this we need knowledge of Calc III, which I know not everyone had, so we will just define it. A convolution is the following operator,

$$(f*g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t f(\tau)g(t-\tau)d\tau. \tag{3}$$

The laplace transform is as folllows,

$$\mathcal{L}\{(f*g)(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}. \tag{4}$$

It should be noted that this is similar to multiplication and has some of the same properties:

$$1) f*g = g*f \qquad 2) f*(g_1+g_2) = f*g_1 + f*g_2 \qquad 3)(f*g)*h = f*(g*h).$$

Now, lets do some problems,

7) We take the laplace transform of sine and cosine and then multiply them together,

$$\mathcal{L}\{\sin t\} = \frac{1}{s^2+1},\ \mathcal{L}\{\cos t\} = \frac{s}{s^2+1} \Rightarrow \mathcal{L}\{f(t)\} = \frac{s}{(s^2+1)^2}$$

11) Here we take the inverse. We know the transform of sine from above and the inverse transform of $G(s)$. So we get,

$$\mathcal{L}^{-1}\{F(s)\} = \int_0^t \sin\tau g(t-\tau)d\tau.$$

17) Here we take the laplace transform of the entire ODE,

$$-\cancelto{-3}{y'(0)}-s\cancelto{2}{y(0)}+s^2Y-4\cancelto{2}{y(0)}+4sY+4Y=G(s)\Rightarrow(s^2+4s+4)Y=2s+5+G(s)\Rightarrow Y=\frac{2s+5}{(s+2)^2}+\frac{G(s)}{(s+2)^2}.$$

We employ partial fractions,

$$\frac{A}{s+2}+\frac{B}{(s+2)^2}=\frac{2s+5}{(s+2)^2}\Rightarrow As+2A+B=2s+5.$$

This gives, $A=2$, $B=1$. Then we get,

$$Y=\frac{2}{s+2}+\frac{1}{(s+2)^2}+\frac{G(s)}{(s+2)^2}.$$

Taking the inverse transform gives,

$$y=2e^{-2t}+te^{-2t}+\int_0^t \tau e^{-2\tau}g(t-\tau)d\tau.$$

16) Again,

$$-\cancelto{-1}{y'(0)}-s\cancelto{1}{y(0)}+s^2Y-\cancelto{1}{y(0)}+sY+\frac{5}{4}Y=\frac{1}{s}-\frac{1}{s}e^{-\pi s}\Rightarrow(s^2+s+5/4)Y=s+\frac{1}{s}-\frac{1}{s}e^{-\pi s}$$

$$\Rightarrow Y=\frac{s}{s^2+s+5/4}+\frac{1-e^{-\pi s}}{s(s^2+s+5/4)}=\frac{s+1/2}{(s+1/2)^2+1}-\frac{1/2}{(s+1/2)^2+1}+\frac{1}{(s+1/2)^2+1}\cdot\frac{1-e^{-\pi s}}{s}$$

$$\Rightarrow y=e^{-t/2}\cos t-\frac{1}{2}e^{-t/2}\sin t+\int_0^t e^{-\tau/2}\sin\tau(1-u_\pi(t-\tau))d\tau.$$

I made a small error in class which changed the problem, so make sure you go over this one.

7.1 Introduction to systems of first order ODEs

In class we went through the example of a simple pendulum. I wont redo that example here, but what we take out of that is the simple pendulum is governed by the ODE: $\theta''+(g/L)\sin\theta=0$. And we can convert this into a system of two first order ODEs by letting $\omega=\theta'$, then $\theta'=\omega$ and $\omega'=-(g/L)\sin\theta$. By doing this we could extract a lot of necessary information to an otherwise unsolvable (with the methods we know) problem. We can use this trick for other problems as done bellow,

1) Let $v=u'$, then $v'=-v/2+2u$.
3) Let $v=u'$, then $v'=-v/t+(1/4-t^2)u/t^2$.
6) Let $v=u'$, then $v'=g(t)-p(t)v-q(t)u$ and $u(0)=u_0$, $v(0)=u_0'$.
10) Notice $x_2=(x_1-x_1')/2$, then

$$((x_1-x_1')/2)'=3x_1-4((x_1-x_1')/2)\Rightarrow x_1'-x_1''=2x_1+4x_1'\Rightarrow x_1''+3x_1'+2x_1=0;\ x(0)=-1,\ x_1'(0)=-5.$$

Now we solve for x_1, $r^2+3r+2=(r+2)(r+1)=0\Rightarrow r=-2,-1$, then

$$x_1=c_1e^{-t}+c_2e^{-2t}.$$

From the initial conditions we get, $x_1(0)=c_1+c_2=-1$ and $x_1'(0)=-c_1-2c_2=-5$, then $c_2=6, c_1=-7$. Now to solve for x_2 we plug x_1 into the first equation where we have x_2 as a function of x_1 and x_1' to get,

$$x_1=6e^{-t}-7e^{-2t};\ x_2=-7e^{-t}+9e^{-2t}.$$