CALC II FORMULAS

$$\text{Integration by parts:} \quad \int u dv = uv - \int v du. \tag{1}$$

$$\text{Taylor series:} \quad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{2}$$

Common Taylor Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots \tag{3}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots \tag{4}$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \tag{5}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots. \tag{6}$$

Partial fractions.

Case 1.

Suppose Q is a product of distinct linear factors, i.e. $Q = (a_1x + b_1)(a_2x + b_2)\cdots(a_kx + b_k)$. Then,

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_k}{a_kx + b_k}. \tag{7}$$

Case 2.

Suppose Q is a product of linear factors, some of which are repeated. Then, the repeated factors are of this form

$$\frac{P(x)}{Q(x)} = \frac{P(x)}{(ax+b)^r} = \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_r}{(ax+b)^r}. \tag{8}$$

Case 3.

Suppose Q is a product of quadratic factors with no repeats, i.e. $Q = (a_1x^2 + b_1x + c_1)(a_2x^2 + b_2x + c_2)\cdots(a_kx^2 + b_kx + c_k)$. Then,

$$\begin{aligned}\frac{P(x)}{Q(x)} &= \frac{P(x)}{(a_1x^2 + b_1x + c_1)(a_2x^2 + b_2x + c_2)\cdots(a_kx^2 + b_kx + c_k)} \\ &= \frac{A_1x + B_1}{a_1x^2 + b_1x + c_1} + \frac{A_2x + B_2}{a_2x^2 + b_2x + c_2} + \cdots + \frac{A_kx + B_k}{a_kx^2 + b_kx + c_k}.\end{aligned} \tag{9}$$

Case 4.

Suppose Q is product of factors that include repeated quadratic factors. Then the repeated quadratic factors will be of the form,

$$\begin{aligned}\frac{P(x)}{Q(x)} = \frac{P(x)}{(ax^2 + bx + c)^r} &= \frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} \\ &+ \cdots + \frac{A_rx + B_r}{(ax^2 + bx + c)^r}.\end{aligned} \tag{10}$$

ODE Formulas and Definitions

Definition 1. The <u>order</u> of an ODE is the order of the highest derivative.

Definition 2. Consider the ODE

$$F(t, y, y', \dots, y^{(n)}) = 0,$$

then the ODE is said to be <u>linear</u> if F is a linear function with respect to $y, y', \dots, y^{(n)}$.

Definition 3. An ODE is <u>separable</u> if it can be written in the form $f(x)dx = g(y)dy$.

$$\text{Integrating Factor: } \frac{dy}{dx} + p(x)y = g(x) \Rightarrow \mu(x) = \exp\left(\int^x p(\xi)d\xi\right) \Rightarrow \mu(x)y = \int \mu(x)g(x)dx. \quad (11)$$

$$\text{Characteristic polynomial: } ay'' + by' + cy = 0 \Rightarrow ar^2 + br + c = 0. \quad (12)$$

$$\text{Distinct roots: } r = r_1, r_2 \Rightarrow y = c_1 e^{r_1 x} + c_2 e^{r_2 x} \quad (13)$$

$$\text{Complex roots: } r = \xi \pm i\theta \Rightarrow y = e^{\xi x}(A\cos\theta x + B\sin\theta x) \quad (14)$$

$$\text{Repeated roots: } r = r \Rightarrow y = (c_1 + c_2 x)e^{rx} \quad (15)$$

$$\text{Wronskian: } W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

Reduction of order
Suppose we know one solution to $y''+p(x)y'+q(x)y = 0$, say y_1. Let $y = u(x)y_1(x)$ and plug into ODE. Then let $v = u'$ and solve first order ODE for v integrate to get u then plug back into $y = u(x)y_1(x) = c_1 y_1 + c_2 y_2$. y_2 is the second solution.

$$\text{Variation of Parameters: } y''+p(x)y'+q(x)y = f(x) \Rightarrow y = -y_1 \int \frac{f(x)y_2}{W(y_1, y_2)}dx + y_2 \int \frac{f(x)y_1}{W(y_1, y_2)}dx. \quad (16)$$

$$\text{Euler's ODE: } ay''(\xi) + by'(\xi) + cy(\xi) = 0 \Rightarrow ar(r-1) + br + c = 0. \quad (17)$$

$$\text{Distinct roots: } r = r_1, r_2 \Rightarrow y = c_1 x^{r_1} + c_2 x^{r_2} \quad (18)$$

$$\text{Complex roots: } r = \xi \pm i\theta \Rightarrow y = x^{\xi}(A\cos(\theta\ln x) + B\sin(\theta\ln x)) \quad (19)$$

$$\text{Repeated roots: } r = r \Rightarrow y = (c_1 + c_2 \ln x)x^r \quad (20)$$

$$\text{Laplace Transform: } F(s) = \int_0^\infty e^{-st} f(t)dt \quad (21)$$

$$\text{Step function: } u_c(t) = \begin{cases} 0, & \text{for } t < c, \\ 1, & \text{for } t \geq c; \end{cases} \quad (22)$$

$$\text{Convolution: } (f * g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t f(\tau)g(t-\tau)d\tau. \quad (23)$$

Undetermined Coefficients

Case1: No term in $f(x)$ is the same as any term in y_c. Then, y_p is a linear combination of terms of $f(x)$ and their derivatives.

Ex: $f_1(x) = x^n \Rightarrow y_{1_p} = A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0$. If our f is a polynomial, the particular solution will be of the form of the most general polynomial of order of that of the polynomial in f.

Ex: $f_2(x) = e^{mx} \Rightarrow y_{2_p} = ke^{mx}$. This one is easy.

Ex: $f_3(x) = \cos(mx)$ or $\sin(mx) \Rightarrow y_{3_p} = A\cos(mx)+B\sin(mx)$. If we have sine or cosine our particular solution will be a linear combination of sines and cosines.

Ex: $f(x) = f_1(x) + f_2(x) + f_3(x) \Rightarrow y_p = y_{1_p} + y_{2_p} + y_{3_p}$. If we have a combination of these simple examples then we just combine all of their respective particular solutions.

Ex: $f(x) = f_1(x)f_2(x)f_3(x) \Rightarrow y_p = y_{1_p}y_{2_p}y_{3_p}$. We do the same sort of thing with products.

Case2: $f(x)$ contains terms that are x^n times terms in y_c, i.e. if $u(x)$ is a term of y_c and $f(x)$ contains $x^n u(x)$. Then y_p is as usual but multiply by "x".

Case3: If y_c contains repeated roots with the highest being of order λ, i.e. x^λ, and $f(x)$ contains terms x^n times the repeated roots terms. Then multiply out by $x^{\lambda+1}$.

Summary of Case 2 and 3:

Case	Characteristic solution	Repeat	Particular solution form
Case2	$y_c = c_1e^{r_1x} + c_2e^{r_2x}$	$f(x) = x^n e^{r_1x}$	$y_p = x(A_nx^n + \cdots + A_1x + A_0)e^{r_1x}$
	$y_c = e^{\xi x}(A\cos(\theta x) + B\sin(\theta x))$	$f(x) = x^n e^{\xi x}\cos(\theta x)$	$y_p = x(A_nx^n + \cdots + A_0)e^{\xi x}\cos(\theta x)$
Case3	$y_c = (c_1 + c_2x)e^{\lambda x}$	$f(x) = x^n e^{\lambda x}$	$y_p = x^2(A_nx^n + \cdots + A_1x + A_0)e^{\lambda x}$

First order 2x2 matrix ODE

Find the two eigenvalues λ_1 and λ_2 then find the corresponding eigenvectors $x^{(1)}$ and $x^{(2)}$ then $x = c_1x^{(1)}e^{\lambda_1 t} + c_2x^{(2)}e^{\lambda_2 t}$.

Boundary Value Problems

Unique solution: Both constants are solved for.

No solution: When solving for the constants one boundary value contradicts the other.

Infinitely many solutions: Only one constant is solved for.

Eigenvalue problem

(i) Solve the problem for $\lambda > 0$ (usually produces a set of eigenvalues and eigenfunctions).

(ii) Solve for $\lambda < 0$ (usually produces trivial solution $y = 0$)

(iii) Solve for $\lambda = 0$ (usually either constant solution $y = 1$ or trivial solution $y = 0$)

Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right], \tag{24}$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx, \; a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx;$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx.$$

Definition 4. Consider a function $f(x)$ such that $f(-x) = f(x)$, then f is said to be even.

Definition 5. Consider a function $f(x)$ such that $f(-x) = -f(x)$, then f is said to be odd.

Properties

- Sum/difference of two even functions is even.
- Sum/difference of two odd functions is odd.
- sum/difference of and even and an odd function is neither even nor odd.
- Product/quotient of two even functions is even.
- Product/quotient of two odd functions is even.
- Product/quotient of an even and an odd function is odd.
- If f is even, $\int_{-L}^{L} f(x)dx = 2\int_0^L f(x)dx$.
- If f is odd, $\int_{-L}^{L} f(x)dx = 0$.

Fourier Cosine Series: If f is an even periodic function generated on $-L \leq x \leq L$, then $b_n = 0$, so

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) \right] \tag{25}$$

$$a_0 = \frac{2}{L}\int_{0}^{L} f(x)dx, \; a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx;$$

Fourier Sine Series: If f is an odd periodic function generated on $-L \leq x \leq L$, then $a_n = 0$, so

$$f(x) = \sum_{n=1}^{\infty} \left[b_n \sin\left(\frac{n\pi x}{L}\right) \right] \tag{26}$$

$$b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx;$$