FALL 2007 SOLUTIONS

(1) (a)

(b) **(i)** For $\lambda > 0$ let $\lambda = \mu^2$, then

$$y = A\cos\mu x + B\sin\mu x \Rightarrow y' = -A\mu\sin\mu x + B\mu\cos\mu x.$$

The first boundary condition gives $y'(0) = B = 0$. The second boundary condition gives, $y'(L) = -A\mu\sin\mu L = 0$, so $\mu = n\pi/L$ for $n = 1, 2, 3, \ldots$. Then our eigenvalues and respective eigenfunctions are

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2;\ y_n = \cos\left(\frac{n\pi x}{L}\right)$$

(ii) For $\lambda < 0$ let $\lambda = -\mu^2$, then

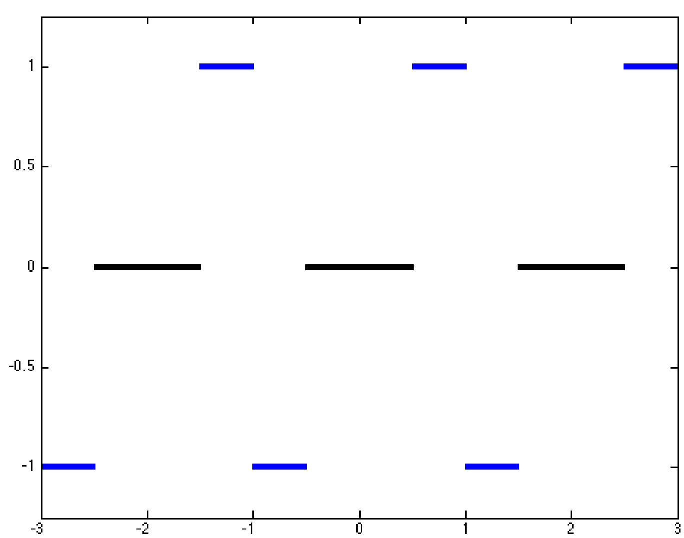
$$y = A\cosh\mu x + B\sinh\mu x \Rightarrow y' = A\mu\sinh\mu x + B\mu\cosh\mu x.$$

The first boundary condition gives, $y'(0) = B = 0$ and the second boundary condition gives, $y'(L) = A\mu\sinh\mu L = 0 \Rightarrow A = 0$, so we get the trivial solution $y = 0$.

(iii) For $\lambda = 0$, $y = c_1 + c_2 x$, so $y'(0) = c_2 = y'(L) = 0 \Rightarrow y = c_1$, so the eigenvalue and eigenvector is,

$$\lambda_0 = 0;\ y_0 = 1.$$

(2) (a) Our sketch should look like the following:



(b) Since they are interested in the Sine series, $a_n = 0$. And we have,

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx = 2\int_0^1 (1-x)\sin(n\pi x)dx.$$

We use integration by parts via $u = 1 - x \Rightarrow du = -dx$ and $dv = \sin(n\pi x)dx \Rightarrow v = (-1/n\pi)\cos(n\pi x)$. Then we get,

$$b_n = \frac{2(x-1)}{n\pi}\cos(n\pi x)\Big|_0^1 - \frac{2}{n\pi}\int_0^1 \cos(n\pi x)dx = \frac{2}{n\pi} - \frac{2}{(n\pi)^2}\cancelto{0}{\sin(n\pi x)\Big|_0^1} = \frac{2}{n\pi}$$

Then our Fourier series is,

$$f(x) = \sum_{n=1}^{\infty}\frac{2}{n\pi}\sin(n\pi x).$$

(3) (a) We first need to manipulate our function,

$$F(s) = \frac{2s+1}{s^2+2s+5} = \frac{2s+1}{(s+1)^2+2^2} = 2\frac{s+1}{(s+1)^2+2^2} - \frac{1}{(s+1)^2+2^2}.$$

The last expression is from formula 9 and 10,

$$f(t) = 2e^{-t}\cos 2t - \frac{1}{2}e^{-t}\sin 2t.$$

(b) Lets define,

$$F(s) = \frac{1-s}{(s+1)^3} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)^3}.$$

From the partial fractions we get,

$$A(s+1)^2 + B(s+1) + C = As^2 + (2A+B)s + (A+B+C) = 1 - s \Rightarrow A = 0 \Rightarrow B = -1 \Rightarrow C = 2.$$

So we get, from formula 11,

$$F(s) = \frac{2}{(s+1)^3} - \frac{1}{(s+1)^2} \Rightarrow f(t) = t^2e^{-t} - te^{-t}.$$

Now we use formula 13,

$$g(t) = u_2(t)f(t-2) = u_2(t)\left[(t-2)^2e^{-(t-2)} - (t-2)e^{-(t-2)}\right].$$

(4) (a) We take the Laplace transform of the entire ODE,

$$(s^2+1)Y = e^{-2\pi s} + \alpha\frac{1}{s}e^{-3\pi s} \Rightarrow Y = \frac{e^{-2\pi s}}{s^2+1} + \alpha\frac{1}{s(s^2+1)}e^{-3\pi s}.$$

Notice,

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1} \Rightarrow Y = \frac{e^{-2\pi s}}{s^2+1} + \alpha\left(\frac{1}{s} - \frac{s}{s^2+1}\right)e^{-3\pi s}.$$

Then taking the inverse transform gives,

$$y = u_{2\pi}(t)\sin(t - 2\pi) + \alpha\left[1 - \cos(t - 3\pi)\right]u_{3\pi}(t).$$

(b) $y(5\pi/2) = \sin(\pi/2) = 1.$

(c) $y(7\pi/2) = \cancelto{-1}{\sin(3\pi/2)} + \alpha - \cancelto{0}{\cos(\pi/2)} = 0 \Rightarrow \alpha = 1.$

(5) (a) This is linear and we must first convert this to standard form,

$$y' + \frac{2}{t}y = \frac{\cos t}{t^2} \Rightarrow \mu = \exp\left(2\int^t \frac{d\tau}{\tau}\right) = t^2 \Rightarrow \mu y = \int \mu g(t)dt$$
$$\Rightarrow t^2 y = \int \cos t dt = \sin t + C \Rightarrow y = \frac{\sin t}{t^2} + \frac{C}{t^2}.$$

Plugging in the initial condition gives, $y(\pi) = C/\pi^2 = 0$, so our solution is,

$$y = \frac{\sin t}{t^2}.$$

(b) This is nonlinear, but we can use separation,

$$\int y dy = \int \frac{2x dx}{1+x^2} \Rightarrow \frac{1}{2}y^2 = \int \frac{du}{u} = \ln|u| + C_1 = \ln(1+x^2) + C_1 \Rightarrow y = \pm\sqrt{\ln(1+x^2)^2 + C_2}.$$

From the initial condition we get $y(0) = \sqrt{C_2} = -2 \Rightarrow C_2 = 4$. Notice this gives us the branch of the root we need to take,

$$y = -\sqrt{\ln(1+x^2)^2 + 4}.$$

(6) (a) We first find the characteristic solution,

$$r^2 + 6r + 13 = 0 \Rightarrow r = \frac{1}{2}\left(-6 \pm \sqrt{36 - 52}\right) = -3 \pm 2i \Rightarrow y_c = e^{-3x}(A\cos 2x + B\sin 2x).$$

We guess the form of the particular solution to be,

$$\tilde{y}_p = e^{-3x}(c_1\cos 2x + c_2\sin 2x) + (D_2x^2 + D_1x + D_0)(E_1\cos 3x + E_2\sin 3x).$$

Notice a repeat with $e^{-3x}(A\cos 2x + B\sin 2x)$, so the form of the particular solution is,

$$y_p = e^{-3x}x(c_1\cos 2x + c_2\sin 2x) + (D_2x^2 + D_1x + D_0)(E_1\cos 3x + E_2\sin 3x).$$

(b) Let $y = uy_1 = ux^2$, then $y' = u'x^2 + 2xu$ and $y'' = u''x^2 + 4xu' + 2u$. Plugging into the ODE gives,

$$x^2\left[u''x^2 + 4xu' + \cancel{2u}\right] - 3x\left[u'x^2 + \cancel{2xu}\right] + \cancel{4ux^2} = x^4u'' + x^3u' = 0.$$

Let $v = u'$, then

$$xv' + v = 0 \Rightarrow \frac{dv}{v} = -\frac{dx}{x} \Rightarrow \ln v = -\ln|x| + C_1 \Rightarrow v = \frac{k_1}{x} \Rightarrow u = k_1\ln|x| + C_2 \Rightarrow y = k_1x^2\ln|x| + C_2x^2.$$

Then our second solution is $y_2 = x^2\ln|x|$.

Spring 2012 Solutions

(1) (a) This is separable so we separate and integrate,

$$\int \frac{dy}{y-4} = \frac{1}{2}\int x dx \Rightarrow \ln(y-4) = \frac{1}{4}x^2 + C \Rightarrow y-4 = ke^{x^2/4} \Rightarrow y = ke^{x^2/4}+4.$$

(b) We take the characteristic polynomial and proceed,

$$r^2 - 6r + 9 = (r-3)^2 = 0 \Rightarrow r = 3, 3 \Rightarrow y = (c_1 + c_2 x)e^{3x}.$$

(c) First we find the characteristic solution,

$$r^2 + 3r + 2 = (r+1)(r+2) = 0 \Rightarrow r = -1, -2 \Rightarrow y_c = c_1 e^{-x} + c_2 e^{-2x}.$$

Next, the form of the is $y_p = Ax + B$, then plugging into the ODE gives,

$$3A + 2Ax + 2B = 4x + 2 \Rightarrow A = 2 \Rightarrow B = -2.$$

Then our solution is,

$$y = y_c + y_p = c_1 e^{-x} + c_2 e^{-2x} + 2x - 2.$$

(2) We take the Laplace transform of the entire ODE,

$$(s^2 - 4s + 13)Y = 4e^{-10s} \Rightarrow Y = \frac{4e^{-10s}}{s^2 - 4s + 13} = \frac{4e^{-10s}}{(s-2)^2 + 3^2} = \frac{4}{3}\frac{3e^{-10s}}{(s-2)^2 + 3^2}$$

$$\Rightarrow y = \frac{4}{3}e^{2(t-10)}\sin(3(t-10))u_{10}(t).$$

(3) Again,

$$(s^2 - 4s + 5)Y = G(s) \Rightarrow Y = \frac{G(s)}{s^2 - 4s + 5} = \frac{G(s)}{(s-2)^2 + 1} \Rightarrow y = \int_0^t e^{2\tau}\sin\tau g(t-\tau)d\tau.$$

(4) We first find the eigenvalues,

$$\begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) - 8 = \lambda^2 - 4\lambda - 5 = 0 \Rightarrow \lambda = \frac{1}{2}(4 \pm \sqrt{16+20}) = 2 \pm 3 = -1, 5.$$

Now we find the respective eigenvectors,

$$x^{(1)} : \begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix} x^{(1)} = 0 \Rightarrow x^{(1)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix};\ x^{(2)} : \begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix} x^{(2)} = 0 \Rightarrow x^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

So our general solution is,

$$x = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{5t}.$$

From the initial condition we get,

$$x(0) = \begin{pmatrix} -c_1 + c_2 \\ c_1 + 2c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow c_2 = 2/3 \Rightarrow c_1 = -1/3.$$

Then our solution is,

$$x = -\frac{1}{3}\begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t} + \frac{2}{3}\begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{5t}.$$

(5) (a) We first find the characteristic solution,

$$r^2 + 9 = 0 \Rightarrow r = \pm 3i \Rightarrow y_c = A\cos 3x + B\sin 3x.$$

Now the particular solution is $y_p = D\cos x + E\sin x$. Plugging this into the ODE gives,

$$-D\cos x + E\sin x + 9D\cos x + 9E\sin x = 8D\cos x + 8E\sin x = \sin x \Rightarrow D = 0, E = \frac{1}{8}.$$

Then our general solution is,

$$y = A\cos 3x + B\sin 3x + \frac{1}{8}\sin x.$$

The first boundary condition gives, $y(0) = A = 0$, then $y' = 3B\cos 3x + (1/8)\sin x$. The second boundary condition gives,

$$y'(\pi) = -3B - \frac{1}{8} = 0 \Rightarrow B = -\frac{1}{24}.$$

Then our solution is,

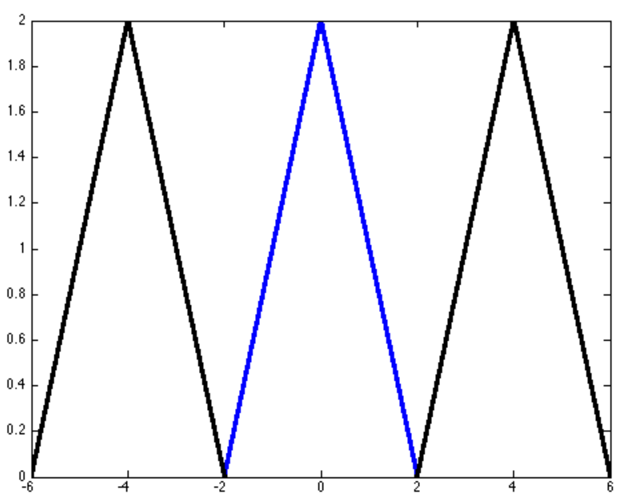
$$y = -\frac{1}{24}\sin 3x + \frac{1}{8}\sin x.$$

(b) $\cos 7x :\ T = 2\pi/7$; $\tan 3x :\ T = \pi/3$; $\sin^2 x :\ T = \pi$.
Now $\sinh 2x$ is a bit harder. This is clearly not periodic but it asks us to prove it. We apply the definition of periodicity,

$$\sinh(2x) = \sin(2(x+T)) = \sinh(2x+2T) = \sinh 2x\cosh 2T + \cosh 2x\sinh 2T.$$

Notice that this only happens when $\cosh 2T = 1$ and $\sinh 2T = 0$, and in both cases $T = 0$, so it is not periodic.

(6) (a) The plot is as follows:



(b) Notice that since $f(x)$ is even $f(x)\sin x$ is odd, so

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi x}{2}\right)dx = 0.$$

Now we calculate a_0 and take advantage of the fact that $f(x)$ is even,

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx = \frac{1}{2}\int_{-2}^{2} f(x)dx = \int_0^2 (2-x)dx = 2x - \frac{x^2}{2}\bigg|_0^2 = 2.$$

Again for a_n,

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right)dx = \int_0^2 (2-x)\cos\left(\frac{n\pi x}{2}\right)dx$$

Now we use by parts with $u = 2 - x \Rightarrow du = -dx$ and $dv = \cos(n\pi x/2)dx \Rightarrow v = (2/n\pi)\sin(n\pi x/2)$,

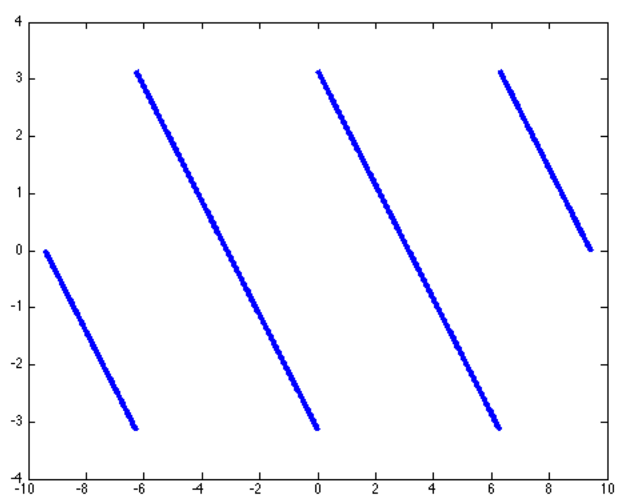
$$\cancelto{0}{\frac{2}{n\pi}(2-x)\sin\left(\frac{n\pi x}{2}\right)\bigg|_0^2} + \int_0^2 \frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)dx = -\left(\frac{2}{n\pi}\right)^2\cos\left(\frac{n\pi x}{2}\right)\bigg|_0^2 = \left(\frac{2}{n\pi}\right)^2 - \left(\frac{2}{n\pi}\right)^2\cos(n\pi)$$

Notice that $\cos(n\pi) = (-1)^n$, so our Fourier series is,

$$f(x) = 1 + \sum_{n=1}^{\infty}\left(\frac{2}{n\pi}\right)^2\left((-1)^{n+1}+1\right)\cos\left(\frac{n\pi x}{2}\right).$$

However, if you wish to stick with $\cos(n\pi)$ instead of converting it that would usually be ok.

(7) (a) The sketch should look as follows:



(b) Since they ask for odd extensions we want the Fourier sine series of the function, so $a_n = 0$, and

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{\pi}\int_0^\pi (\pi - x)\sin nx dx.$$

We use by parts with $u = \pi - x \Rightarrow du = -dx$ and $dv = \sin x dx \Rightarrow v = -(1/n)\cos nx$,

$$b_n = \frac{2(x-\pi)}{n\pi}\cos nx\Big|_0^\pi - \int_0^\pi \frac{2}{n\pi}\cos nx dx = \frac{2}{n} + \cancelto{0}{\frac{2}{n^2\pi}\sin nx\Big|_0^\pi}.$$

So the Fourier sine series for our function is,

$$f(x) = \sum_{n=1}^\infty \frac{2}{n}\sin nx.$$