2.3 The Inverse of a Matrix

In linear algebra we want to find a matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$. We can only find an inverse if A is nonsingular, so invertible and nonsingular mean the same thing, and noninvertible and singular mean the same thing. If A can be pupt into upper triangular form, it is nonsingular, otherwise it is singular.

Note: nonsquare matrices do not have inverses, but we can have either a right or left hand inverse, which we will talk about later.

Now lets do an example where we find the inverse of a matrix. Once more we use our usual 3×3 matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \tag{1}$$

We append the identity matrix to A and use Gauss-Jordan elimination.

$$\begin{matrix} \\ 2 \\ -1 \end{matrix}\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 4 & -6 & 0 & 0 & 1 & 0 \\ -2 & 7 & 2 & 0 & 0 & 1 \end{array}\right] = \begin{matrix} \\ \\ -1 \end{matrix}\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 8 & 3 & 1 & 0 & 1 \end{array}\right] = \begin{matrix} 1 \\ -2 \\ \\ \end{matrix}\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$$= \begin{matrix} -1/8 \\ \\ \\ \end{matrix}\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 2 & -1 & -1 \\ 0 & -8 & 0 & -4 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] = \begin{matrix} 1/2 \\ -1/8 \\ \\ \end{matrix}\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 12/8 & -5/8 & -6/8 \\ 0 & -8 & 0 & -4 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 12/16 & -5/16 & -6/16 \\ 0 & 1 & 0 & 4/8 & -3/8 & -2/8 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 12/16 & -5/16 & -6/16 \\ 4/8 & -3/8 & -2/8 \\ -1 & 1 & 1 \end{bmatrix}$$

Now lets do some problems from the book.

7) This one is easy since there are only diagonal terms, so

$$\begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}$$

9) Here we actually have to do all of Gauss-Jordan

$$\begin{matrix} \\ 3 \end{matrix}\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 7 & 0 & 1 \end{array}\right] = \begin{matrix} 2 \\ \\ \end{matrix}\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & -3 & 1 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & 7 & -2 \\ 0 & 1 & -3 & 1 \end{array}\right]$$

Don't be fooled by the simplicity of this answer though. Even for 2×2, the pattern may not be as you see here. This one is a special case since the determinant (which we will cover later) is 1.

15) Notice that the 3rd row is 2 times the second plus the first, and hence will be eliminated; i.e., a row of zeros. Therefore, it is singular (noninvertible).

17)

$$\begin{matrix} \\ 3 \\ -2 \end{matrix}\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 1 & 0 \\ -2 & 0 & 3 & 0 & 0 & 1 \end{array}\right]$$