## Sec. 3.1 and 4.1 Inner products and orthogonality

We briefly reviewed how to do dot products. Then we talked about orthogonality and projections.

Brief dot product review

You would have seen this in previous courses, especially Calc III

- Length of a vector: $\|v\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} = \sqrt{v \cdot v}$.
- A unit vector is a vector that has length one. Any vector $v$ can be turned into a unit vector by dividing by its length: $v/\|v\|$. This type of vector is also said to be normal.
- Distance between vectors: $\|u - v\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2}$.
- Angle between vectors (law of cosines): $\cos\theta = u \cdot v/\|u\|\|v\|$. Notice that a right angle $\theta = \pi/2$ implies $u \cdot v = 0$, so this is another way that we can show two vectors are perpendicular.
- Pythagorean theorem: $\|u + v\|^2 = \|u\|^2 + \|v\|^2$.

**Definition 1.** Consider $\vec{u}, \vec{v}, \vec{w} \in V \subseteq \mathbb{R}^n$. The product $\langle \vec{v}, \vec{w}\rangle \in \mathbb{R}$ is said to be an inner product if for scalars $c, d \in \mathbb{R}$,

(i) $\langle c\vec{u} + d\vec{v}, \vec{w}\rangle = c\langle \vec{u}, \vec{w}\rangle + d\langle \vec{v}, \vec{w}\rangle$, and
$\langle \vec{u}, c\vec{v} + d\vec{w}\rangle = c\langle \vec{u}, \vec{v}\rangle + d\langle \vec{u}, \vec{w}\rangle$.

(ii) $\langle \vec{v}, \vec{w}\rangle = \langle \vec{w}, \vec{v}\rangle$

(iii) $\langle \vec{v}, \vec{v}\rangle > 0$ whenever $\vec{v} \neq \vec{0}$, and $\langle \vec{v}, \vec{v}\rangle = 0$ only if $\vec{v} = \vec{0}$.

A dot product is a type of inner product, and that will be the only inner product we will use for now.

**Definition 2.** If $\langle \vec{v}, \vec{w}\rangle = 0$, $\vec{v}$ and $\vec{w}$ are said to be orthogonal.

For vectors in $\mathbb{R}^n$ this corresponds to $\vec{v} \cdot \vec{w}$ and $\vec{v}$ and $\vec{w}$ are said to be perpendicular.

Orhtogonality

We notice that right angles are the most important angles in linear algebra. Recall that the four fundamental subspaces meet at right angles. Also our standard bases are produced using orthogonal vectors, in fact they are even better; they are unit vectors, which is referred to as normal. So, $\hat{i}, \hat{j}, \hat{k}$ are orthonormal.

Ex: The following vectors are orthogonal.

$$\begin{pmatrix} -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 2 \end{pmatrix} = 0.$$

Ex: Same as above

$$\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} = 0.$$

**Theorem 1.** *If nonzero vectors $v_1, \ldots, v_k$ are mutually orthogonal (every vector is perpendicular to every other vector), then those vectors are linearly independent.*

An example of mutually orthogonal vectors are the standard basis vectors: $\hat{i}$, $\hat{j}$, $\hat{k}$ in $\mathbb{R}^3$. They are clearly linearly independent. Notice that they are also normal.

There are times when orthogonal will not mean exactly right angles, such as when we look at functions:

$$\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \cdot \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

Notice that these vectors are also normal since $\cos^2\theta + \sin^2\theta = 1$. However, these vector functions are not in euclidean space, but rather in function space, so our concept of perpendicular does not hold for these.

**Definition 3.** A basis is an orthonomral basis if it consists of mutually orthogonal unit vectors; i.e., perpendicular vectors of length one.

Also, we notice that we can only have certain combinations of orthogonal vectors in a finite subspace. Take $\mathbb{R}^3$ for example. We can only have two lines or a line and a plane, but we cannot have two planes orthogonal to each other.

Ex: In $\mathbb{R}^4$ suppose $V$ is the plane spanned by the vectors $v_1 = (1, 0, 0, 0)$ and $v_2 = (1, 1, 0, 0)$. If $W$ is the line spanned by $w = (0, 0, 4, 5)$, then $w$ is orthogonal to both $v_1$ and $v_2$. So, the subspaces $W$ and $V$ are mutually orthogonal.

**Theorem 2** (orthogonality)**.** *The row space $\mathcal{C}(A^T)$ and the nullspace $\mathcal{N}(A)$ are orthogonal to each other, as are the column space $\mathcal{C}(A)$ and the left nullspace $\mathcal{N}(A^T)$.*

*Proof.* Notice that $Ax = 0$ for $\mathcal{N}(A)$, but the nonzero rows of $A$ make up $\mathcal{C}(A^T)$. So, each $x \in \mathcal{N}(A)$ is orthogonal to each row of $A$. □