Sec. 1.9 Determinants

The determinant is a way to measure the "size" of a matrix. Think of it as the analog of absolute value in real numbers, or the modulus in complex numbers. Lets look at examples of 2×2 and 3×3 matrices. These can be easily extended to $n \times n$.

2×2

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} \tag{1}$$

3×3

For higher order determinants such as 3×3, we use cofactor expansion. While there are some "shortcuts", I personally don't see them saving much time.

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{13} & a_{32} & a_{33} \end{pmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{13} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \tag{2}$$

Then just take the determinants of each 2×2 matrix and you are done! For a 4×4 matrix you do the same thing, but you would get 3×3 cofactors. We did a generic example in the lecture.

Now lets look at some examples

Ex:

$$\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 8 - 3 = 5.$$

Ex:

$$\begin{vmatrix} 5 & 2 \\ -6 & 3 \end{vmatrix} = 15 + 12 = 27$$

Ex:

$$\begin{vmatrix} 1 & 4 & -2 \\ 3 & 2 & 0 \\ -1 & 4 & 3 \end{vmatrix} = 1 \begin{vmatrix} 2 & 0 \\ 4 & 3 \end{vmatrix} - 4 \begin{vmatrix} 3 & 0 \\ -1 & 3 \end{vmatrix} + (-2) \begin{vmatrix} 3 & 2 \\ -1 & 4 \end{vmatrix} = -58$$

Ex:

$$\begin{vmatrix} 2 & 4 & 6 \\ 0 & 3 & 1 \\ 0 & 0 & -5 \end{vmatrix} = 2 \begin{vmatrix} 3 & 1 \\ 0 & -5 \end{vmatrix} = 2 \cdot 3 \cdot (-5).$$

Notice that ince we have an upper triangular matrix, all we need to do is take the product of the diagonal: $\det = 2 \cdot 3 \cdot (-5) = -30$

Properties of Determinants

We will list these generically, but illustrate with 2×2 matrices.

(1) $\det(I) = 1$.

E.g.,

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1; \qquad \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1; \qquad \text{etc.}$$

(2) Row exchange changes sign of the determinant.

E.g.,

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = -(bc - ad) = - \begin{vmatrix} c & d \\ a & b \end{vmatrix}$$

(3) If two rows of A are equal, then $\det(A) = 0$

E.g.,

$$\begin{vmatrix} a & b \\ a & b \end{vmatrix} = ab - ba = 0.$$

(4) If A has a row (or column) of zeros, then $\det(A) = 0$.

$$\begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix} = 0; \qquad \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix} = 0.$$

(5) If $A_{n\times n}$ is triangular (either lower or upper) then $\det(A) = a_{11}a_{22}a_{33}\cdots a_{nn}$; i.e., take the product of the diagonal elements.

E.g.,

$$\begin{vmatrix} a & 0 \\ c & d \end{vmatrix} = ad; \qquad \begin{vmatrix} a & b \\ 0 & d \end{vmatrix} = ad.$$

Can we prove this for $n \times n$ matrices?

Handwavey proof. Consider,

$$D = \begin{bmatrix} a_{11} & \cdots & \cdots & \cdots & \cdots \\ & a_{22} & \cdots & \cdots & \cdots \\ & & \ddots & & \\ & \text{\Large 0} & & \ddots & \\ & & & & a_{nn} \end{bmatrix}$$

Then

$$D = \begin{vmatrix} a_{11} & \cdots & \cdots & \cdots & \cdots \\ & a_{22} & \cdots & \cdots & \cdots \\ & & \ddots & & \\ & \text{\Large 0} & & \ddots & \\ & & & & a_{nn} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & \cdots & \cdots & \cdots & \cdots \\ & a_{33} & \cdots & \cdots & \cdots \\ & & \ddots & & \\ & \text{\Large 0} & & \ddots & \\ & & & & a_{nn} \end{vmatrix} = a_{11}a_{22} \begin{vmatrix} a_{33} & \cdots & \cdots & \cdots & \cdots \\ & a_{44} & \cdots & \cdots & \cdots \\ & & \ddots & & \\ & \text{\Large 0} & & \ddots & \\ & & & & a_{nn} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} \begin{vmatrix} a_{44} & \cdots & \cdots & \cdots & \cdots \\ & a_{55} & \cdots & \cdots & \cdots \\ & & \ddots & & \\ & \text{\Large 0} & & \ddots & \\ & & & & a_{nn} \end{vmatrix} = \cdots = a_{11}a_{22}a_{33}\cdots a_{(n-1)(n-1)}a_{nn}.$$

□

(6) If A is singular, then $\det(A) = 0$; otherwise $\det(A) \neq 0$, then A is invertible. If A is in row-echelon form (not to be confused with the full augmented matrix), but not upper triangular, then at least one of the pivots will be zero.

(7) Product rule: $|A||B| = |AB|$.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} e & f \\ g & h \end{vmatrix} = (ad - bc)(eh - fg) = (ae + bg)(cf + dh) - (af + bh)(ce + dg) = \begin{vmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{vmatrix}$$

However, for scalar multiplication, $|cD_{n\times n}| = c^n |D_{n\times n}|$ because this can be written as

$$cD_{n\times n} = \begin{bmatrix} c & & & \\ & c & \text{\Large 0} & \\ \text{\Large 0} & & \ddots & \\ & & & c \end{bmatrix}$$

A special case is $|A||A^{-1}| = |AA^{-1}| = |I| = 1 \Rightarrow |A^{-1}| = 1/|A|$.

(8) Transpose: $|A| = |A^T|$.

E.g.,

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = |A^T|.$$