Week 8 Part 2: Separable ODEs

Separable equations are the easiest equations to solve. This why it's extremely important to recognize separable equations. It will save you a lot of work! One thing we will notice right away is that Autonomous first order ODEs are always separable.

Definition 1. An ODE is separable if it can be written in the form $f(x)dx = g(y)dy$.

For the next few problems we will solve some separable equations.

Ex: $y' = x^2/y$

Solution: We separate the equation by "moving" y to the left and dx to the right,

$$ydy = x^2dx \Rightarrow \frac{1}{2}y^2 = \frac{1}{3}x^3 + C_0 \Rightarrow y = \pm\sqrt{\frac{2}{3}x^3 + C_1};\ y \neq 0.$$

Ex: $y' = \dfrac{3x^2-1}{3+2y}$

Solution: We separate the equation by moving $3+2y$ to the left and dx to the right,

$$(3+2y)dy = (3x^2-1)dx \Rightarrow 3y + y^2 = x^3 - x + C;\ y \neq -\frac{3}{2}.$$

Ex: $y' = (1-2x)/y$

Solution: This type of problem is called an "Initial Value Problem" (IVP). The idea is to use the Initial Value to solve for the constant of integration. First we separate the problem by moving y to the left and dx to the right,

$$ydy = (1-2x)dx \Rightarrow \frac{1}{2}y^2 = x - x^2 + C.$$

Now, the initial value tells us that $y = -2$ when $x = 1$, so if we plug this into the above equation we get that $C = 2$, so plugging it back in and solving for y gives,

$$y = -\sqrt{2x - 2x^2 + 4};\ y \neq 0.$$

Notice we only chose the negative branch of the root because the initial condition starts with negative for the y value and we know that $y \neq 0$ so the solution can't magically cross into the positive branch, so we must stay on the negative branch for all time.

For part b and c, we did the plot in class, and the domain of existence is $-1 < x < 2$.

Ex: $\dfrac{dy}{dt} = ty\dfrac{4-y}{1+t}$

Solution: Separating gives us,

$$\int \frac{dy}{y(4-y)} = \int \frac{tdt}{1+t} \Rightarrow \frac{1}{4}\int\left(\frac{1}{y} + \frac{1}{4-y}\right)dy = \int \frac{u-1}{u}du \Rightarrow \frac{1}{4}\left[\ln|y| - \ln|4-y|\right] = u - \ln|u| + C_0$$

$$\Rightarrow \ln|\frac{y}{4-y}| = 4t - 4\ln|1+t| + C_1 \Rightarrow \frac{y}{4-y} = e^{4t}\frac{K}{(1+t)^4};$$

plugging in the initial condition gives us $K = y_0/(4-y_0)$, then the full solution is

$$y = \frac{\frac{4e^{4t}y_0}{(1+t)^4(4-y_0)}}{1 + \frac{e^{4t}y_0}{(1+t)^4(4-y_0)}} \tag{1}$$

(a) As $t \to \infty$, $y \to 4$.
(b) $y_0 = 2 \Rightarrow K = 1$, so when $y = 3.99$, $T \approx 2.84367$ (via wolfram).
(c) Now, if $t = 2$, $y/(4-y) = e^8e^{-4\ln 3}K$, then $y = 3.99 \Rightarrow y_0 \approx 3.6622$ and $y = 4.1 \Rightarrow y_0 \approx 4.4042$. This gives us an interval of $3.6622 \leq y_0 \leq 4.4042$.