Week 5: Root Finding

For these notes you may want to read them with the lecture videos or coding lecture. I show the illustrations in the lecture video, and do a more concrete example in the coding lecture.

This may seem silly because we have been finding roots since like 6^{th} grade. However, recall that we can only do this for low order polynomials. Further, we need to find roots of the characteristic polynomial to analyze/solve differential equations. Plus there are other applications for this.

Let's consider a function similar to $f(x) = x^3$.

Section search (iterate and pray):

Consider the interval $[a, b]$ such that $f(a) < 0$ and $f(b) > 0$, so by the intermediate value theorem we know there is some point $x = \xi \in (a, b)$ such that $f(\xi) = 0$. We just want to approximate the root. One thing we can do is break up $[a, b]$ into tiny intervals, and see which $f(x_n)$ is closest to zero. Consider

$$[a, b] = [a, x_1] \cup [x_1, x_2] \cup \cdots \cup [x_n, x_{n+1}] \cup \cdots \cup [x_N, b].$$

Then test each of the nodes, $f(x_1), f(x_2), \ldots, f(x_n), \ldots, f(x_N)$. Notice that this is quite slow.

Bisection:

Since we know the root is between a and b, again by the intermediate value theorem we can just test the mid point: $x_1 = (a + b)/2$. If $f(x_1) < 0$, we will redefine $a = x_1$, but if $f(x_1) > 0$, we will redefine $b = x_1$. We repeat this until we get the desired error. The nice thing about bisection is that it will definitely converge as long as $f(a)$ and $f(b)$ have different signs; i.e., there is a root between a and b.

Newton's method:

Suppose we know that there is a root near x_0. Lets find the tangent line of $f(x)$ at x_0 and record its intersection with the x-axis. We repeat this process.

Newton's method also has a general formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

This is derived directly from the equation of a line. Consider two points on a line (x_1, y_1) and (x_2, y_2) (for us one point will be on the curve $f(x)$ and the other point will be on the x-axis). Further, recall that the slope of the tangent line is just the derivative of the function at that point. Then

$$m = \frac{y_1 - y_2}{x_1 - x_2} \Rightarrow f'(x_n) = \frac{f(x_n) - 0}{x_n - x_{n+1}} \Rightarrow (x_n - x_{n+1})f'(x_n) = f(x_n) \Rightarrow x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

While it converges very fast, if it does converge, there are quite a few draw backs:

- Calculating $f'(x)$ may be prohibitive,
- It may not converge (for example if there are many nearby roots),
- It converges slowly for multiplicities.