13.6 Nonhomogeneous Heat Conduction Examples

Ex: Consider the following nonhomogeneous boundary condition problem

$$u_t = ku_{xx}; \qquad u(0,t) = A, \quad u(L,t) = B; \qquad u(x,0) = f(x). \tag{1}$$

We first look for the easiest solution: the equilibrium temperature. What does equilibrium mean? We solve the problem

$$\frac{\partial u_*}{\partial t} = 0 \Rightarrow \frac{\partial^2 u_*}{\partial x^2} = 0; \qquad u_*(0) = A, \quad u_*(L) = B.$$

So, $u_* = c_1 x + c_2$, and $u_*(0) = c_2 = A$, $u_*(L) = c_1 L + A = B$, then our equilibrium solution is $u_* = \frac{B-A}{L}x + A$. Obviously, this does not solve the problem, but it does allow us to make a change of variables that makes the B.C.'s homogeneous. Let $v(x,t) = u(x,t) - u_*(x)$. Taking a time derivative kills u_* and taking two spatial derivatives also kills u_*, so we get

$$v_t = kv_{xx}; \qquad v(0,t) = v(L,t) = 0; \qquad v(x,0) = f(x) - u_* = f(x) - \frac{B-A}{L}x + A \tag{2}$$

We know

$$v(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} e^{-k(n\pi/L)^2 t}, \tag{3}$$

then

$$v(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} = f(x) - \frac{B-A}{L}x + A$$
$$\Rightarrow A_n = \frac{2}{L}\int_0^L (f(x) - \frac{B-A}{L}x + A)\sin\frac{n\pi x}{L}dx$$

which gives us

$$u(x,t) = \frac{B-A}{L}x + A + \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} e^{-k(n\pi/L)^2 t} \tag{4}$$

Ex: Now lets look at an example where the PDE itself is nonhomogeneous

$$u_t = ku_{xx} + Q; \qquad u(0,t) = A, \quad u(L,t) = B; \qquad u(x,0) = f(x). \tag{5}$$

Then $u_{xx} = -Q/k \Rightarrow u_* = -Qx^2/2k + c_1 x + c_2$. Plugging in the BCs gives us $u_*(0) = c_1 = A$ and

$$u_*(L) = -\frac{Q}{2k}L^2 + c_1 L + A = B \Rightarrow c_1 = \frac{1}{L}\left[B - A + \frac{Q}{2k}L^2\right] \Rightarrow u_* = -\frac{Q}{2k}x^2 + \frac{x}{L}\left[B - A + \frac{Q}{2k}L^2\right] + A$$

Letting $v(x,t) = u(x,t) - u_*(x)$ gives us our homogenized equation.

4)

$$u_t = ku_{xx}; \qquad u(0,t) = u_0, \quad u(1,t) = u_1; \qquad u(x,0) = f(x) \tag{6}$$

Solution: $u_{xx} = 0 \Rightarrow u_* = c_1 x + c_2$, so $u_*(0) = c_2 = u_0$ and $u_*(1) = c_1 + u_0 = u_1 \Rightarrow c_1 = u_1 - u_0$, then our equilibrium solution is $u_* = (u_1 - u_0)x + u_0$. Letting $v = u - u_*$ gives us our homogenized equation.

6) This problem is a bit harder and won't come up on any exam, but I keep it here for the interested reader.

$$u_t = ku_{xx} - hu; \qquad u(0,t) = 0, \quad u(\pi,t) = u_0; \qquad u(x,0) = 0. \tag{7}$$

Solution: To get the equilibrium solution we solve

$$ku_{xx} - hu = 0 \Rightarrow u_* = C_1 \cosh\left(\sqrt{\frac{h}{k}}x\right) + C_2 \sinh\left(\sqrt{\frac{h}{k}}x\right)$$

The first boundary gives us $u_*(0)C_1 = 0$ and the second gives us

$$u_*(\pi) = C_2 \sinh\left(\sqrt{\frac{h}{k}}\pi\right) = u_0 \Rightarrow C_2 = \frac{u_0}{\sinh\left(\sqrt{\frac{h}{k}}\pi\right)} \Rightarrow u_* = u_0 \frac{\sinh\left(\sqrt{\frac{h}{k}}x\right)}{\sinh\left(\sqrt{\frac{h}{k}}\pi\right)}$$

Letting $v(x,t) = u(x,t) - u_*(x)$ gives us

$$v_t = kv_{xx} - hv; \qquad v(0,t) = v(\pi,t) = 0; \qquad v(x,0) = -u_0 \frac{\sinh\left(\sqrt{\frac{h}{k}}x\right)}{\sinh\left(\sqrt{\frac{h}{k}}\pi\right)} \tag{8}$$