13.5 Laplace Equation for Steady State Heat Conduction Examples

Ex: Consider the full Laplace problem with prescribed heat at the boundaries.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} = 0; \qquad u(0,y) = g_1(y), \quad u(L,y) = g_2(y); \qquad u(x,0) = f_1(x), \quad u(x,H) = f_2(x). \tag{1}$$

Uh oh, we don't have homogeneous boundary conditions anymore. So, we can't use straight separation. What does the principle of super position tell us? Let us first solve in one direction and then the other direction and add the two solutions. This will allow us to make one direction homogeneous first and then the next.

Problem 1: Let us first work in the x direction.

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial y^2} = 0; \qquad v(x,0) = f_1(x), \quad v(x,H) = f_2(x); \qquad v(0,y) = v(L,y) = 0. \tag{2}$$

Let $v = X(x)Y(y) \Rightarrow X''Y + XY'' = 0 \Rightarrow X''/X = -Y''/Y$. Now we have a decision to make: which problem is Sturm–Liouville? The one with the homogeneous boundary conditions will be, so let $X''/X = -Y''/Y = -\lambda^2$.

For $\lambda = 0$ we get $X = c_1 x + c_2$, then $X(0) = c_2 = 0$ and $X(L) = Lc_1 = 0$. And for $\lambda \neq 0.$,

$$X = C_1 \cos \lambda x + C_2 \sin \lambda x, \qquad Y = D_1 \cosh \lambda y + D_2 \sinh \lambda y$$

Recall that with homogeneous boundary conditions sinh and cosh give trivial solutions, which is why we write λ^2 from the start, but for Y the boundary conditions are nonhomogeneous, so sinh and cosh are fair game.

Now we solve the homogeneous part to get our eigenvalue

$$X(0) = C_1 = 0, \qquad X(L) = C_2 \sin \lambda L = 0 \Rightarrow \lambda = \frac{n\pi}{L}.$$

Then we get the general solution

$$v(x,y) = \sum_{n=1}^{\infty} A_n \cosh\frac{n\pi y}{L} \sin\frac{n\pi x}{L} + B_n \sinh\frac{n\pi y}{L} \sin\frac{n\pi x}{L} \tag{3}$$

Plugging in our other boundary conditions gives us

$$v(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} = f_1(x) \Rightarrow A_n = \frac{2}{L}\int_0^L f_1(x) \sin\frac{n\pi x}{L} dx$$

and

$$v(x,H) = \sum_{n=1}^{\infty} A_n \cosh\frac{n\pi H}{L} \sin\frac{n\pi x}{L} + B_n \sinh\frac{n\pi H}{L} \sin\frac{n\pi x}{L} = f_2(x)$$

Notice that we can pull out a sin and consider one big constant for our Fourier Series integrals

$$\sum_{n=1}^{\infty} \left[A_n \cosh\frac{n\pi H}{L} + B_n \sinh\frac{n\pi H}{L}\right] \sin\frac{n\pi x}{L} = f_2(x)$$

$$\Rightarrow A_n \cosh\frac{n\pi H}{L} + B_n \sinh\frac{n\pi H}{L} = \frac{2}{L}\int_0^L f_2(x) \sin\frac{n\pi x}{L} dx$$

$$\Rightarrow B_n = \frac{\frac{2}{L}\int_0^L f_2(x) \sin\frac{n\pi x}{L} dx - A_n \cosh\frac{n\pi H}{L}}{\sinh\frac{n\pi H}{L}}$$

Then we put it all together to get

$$v(x,y) = \sum_{n=1}^{\infty} \frac{2}{L}\left[\int_0^L f_1(x)\sin\frac{n\pi x}{L}dx\right]\cosh\frac{n\pi y}{L}\sin\frac{n\pi x}{L} + \left[\frac{\frac{2}{L}\int_0^L f_2(x)\sin\frac{n\pi x}{L}dx - \cosh\frac{n\pi H}{L}\frac{2}{L}\int_0^L f_1(x)\sin\frac{n\pi x}{L}dx}{\sinh\frac{n\pi H}{L}}\right]\sinh\frac{n\pi y}{L}\sin\frac{n\pi x}{L} \quad (4)$$

Problem 2: Now we look at the other direction.

$$\frac{\partial^2 w}{\partial w^2} + \frac{\partial w}{\partial w^2} = 0; \qquad w(x,0) = w(x,H) = 0; \qquad w(0,y) = g_1(y), \quad w(L,y) = g_2(y). \quad (5)$$

Let $v = X(x)Y(y) \Rightarrow X''Y + XY'' = 0 \Rightarrow X''/X = -Y''/Y$. Now we have a decision to make: which problem is Sturm–Liouville? The one with the homogeneous boundary conditions will be, so let $X''/X = -Y''/Y = \lambda^2$.

For $\lambda = 0$ we get $Y = c_1 y + c_2$, then $Y(0) = c_2 = 0$ and $Y(H) = Hc_1 = 0$. And for $\lambda \neq 0.$,

$$Y = C_1\cos\lambda y + C_2\sin\lambda y, \qquad X = D_1\cosh\lambda x + D_2\sinh\lambda x$$

Recall that with homogeneous boundary conditions sinh and cosh give trivial solutions, which is why we write λ^2 from the start, but for Y the boundary conditions are nonhomogeneous, so sinh and cosh are fair game.

Now we solve the homogeneous part to get our eigenvalue

$$Y(0) = C_1 = 0, \qquad Y(H) = C_2\sin\lambda H = 0 \Rightarrow \lambda = \frac{n\pi}{H}.$$

Then we get the general solution

$$w(x,y) = \sum_{n=1}^{\infty} A_n\cosh\frac{n\pi x}{H}\sin\frac{n\pi y}{H} + B_n\sinh\frac{n\pi x}{H}\sin\frac{n\pi y}{H} \quad (6)$$

Plugging in our other boundary conditions gives us

$$w(0,y) = \sum_{n=1}^{\infty} A_n\sin\frac{n\pi y}{H} = g_1(y) \Rightarrow A_n = \frac{2}{H}\int_0^H g_1(y)\sin\frac{n\pi y}{H}dy$$

and

$$w(L,y) = \sum_{n=1}^{\infty} A_n\cosh\frac{n\pi L}{H}\sin\frac{n\pi y}{H} + B_n\sinh\frac{n\pi L}{H}\sin\frac{n\pi y}{H} = g_2(y)$$

Notice that we can pull out a sin and consider one big constant for our Fourier Series integrals

$$\sum_{n=1}^{\infty}\left[A_n\cosh\frac{n\pi L}{H} + B_n\sinh\frac{n\pi L}{H}\right]\sin\frac{n\pi y}{H} = g_2(y)$$

$$\Rightarrow A_n\cosh\frac{n\pi L}{H} + B_n\sinh\frac{n\pi L}{H} = \frac{2}{H}\int_0^H g_2(y)\sin\frac{n\pi y}{H}dx$$

$$\Rightarrow B_n = \frac{\frac{2}{H}\int_0^H g_2(y)\sin\frac{n\pi y}{H}dy - A_n\cosh\frac{n\pi L}{H}}{\sinh\frac{n\pi L}{H}}$$

Then we put it all together to get

$$w(x,y) = \sum_{n=1}^{\infty} \frac{2}{H}\left[\int_0^H g_1(y)\sin\frac{n\pi y}{H}dy\right]\cosh\frac{n\pi x}{H}\sin\frac{n\pi y}{H} + \left[\frac{\frac{2}{H}\int_0^H g_2(y)\sin\frac{n\pi y}{H}dy - \cosh\frac{n\pi L}{H}\frac{2}{H}\int_0^H g_1(y)\sin\frac{n\pi y}{H}dy}{\sinh\frac{n\pi L}{H}}\right]\sinh\frac{n\pi x}{H}\sin\frac{n\pi y}{H} \quad (7)$$

So, the complete solution is

$$
\begin{aligned}
u(x,y) &= v(x,y) + w(x,y) \\
&= \sum_{n=1}^{\infty} \frac{2}{L}\left[\int_0^L f_1(x)\sin\frac{n\pi x}{L}dx\right]\cosh\frac{n\pi y}{L}\sin\frac{n\pi x}{L} \\
&\quad + \left[\frac{\frac{2}{L}\int_0^L f_2(x)\sin\frac{n\pi x}{L}dx - \cosh\frac{n\pi H}{L}\frac{2}{L}\int_0^L f_1(x)\sin\frac{n\pi x}{L}dx}{\sinh\frac{n\pi H}{L}}\right]\sinh\frac{n\pi y}{L}\sin\frac{n\pi x}{L} \\
&\quad + \frac{2}{H}\left[\int_0^H g_1(y)\sin\frac{n\pi y}{H}dy\right]\cosh\frac{n\pi x}{H}\sin\frac{n\pi y}{H} \\
&\quad + \left[\frac{\frac{2}{H}\int_0^H g_2(y)\sin\frac{n\pi y}{H}dy - \cosh\frac{n\pi L}{H}\frac{2}{H}\int_0^H g_1(y)\sin\frac{n\pi y}{H}dy}{\sinh\frac{n\pi L}{H}}\right]\sinh\frac{n\pi x}{H}\sin\frac{n\pi y}{H}
\end{aligned}
\tag{8}
$$

7) Here is a problem from the book

$$
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0; \qquad \left.\frac{\partial u}{\partial x}\right|_{x=0} = u(0,y), \quad u(\pi,y) = 1; \qquad u(x,0) = u(x,\pi) = 0 \tag{9}
$$

Solution: Let $u = XY \Rightarrow X''Y + XY'' = 0 \Rightarrow X''/X = -Y''/Y = \lambda^2$. Notice we choose $+\lambda^2$ instead of $-\lambda^2$ because our Sturm-Liouville problem is in the y direction.

This gives us the ODEs

$$
X'' - \lambda^2 X = 0 \qquad Y'' + \lambda^2 Y = 0. \tag{10}
$$

For $\lambda = 0$ we get $Y = c_1 y + c_2$, so $Y(0) = c_2 = 0$ and $Y(\pi) = c_1\pi = 0$. And for $\lambda \neq 0$

$$
Y = C_1\cos\lambda y + C_2\sin\lambda y, \qquad X = D_1\cosh\lambda x + D_2\sinh\lambda x
$$

Then $Y(0) = C_1 = 0$ and $Y(\pi) = C_2\sin\lambda\pi = 0 \Rightarrow \lambda = n$, so $Y = C_2\sin(ny)$. The our general solution is

$$
u(x,y) = \sum_{n=1}^{\infty} A_n\cosh(nx)\sin(ny) + B_n\sinh(nx)\sin(ny). \tag{11}
$$

Lets solve the boundary at zero first,

$$
\begin{aligned}
\frac{\partial u}{\partial x} &= \sum_{n=1}^{\infty} nA_n\sinh(nx)\sin(ny) + nB_n\cosh(nx)\sin(ny) \\
\left.\frac{\partial u}{\partial x}\right|_{x=0} &= \sum_{n=1}^{\infty} nB_n\sin(ny)
\end{aligned}
$$

And $u(0,y) = \sum_{n=0}^{\infty} A_n\sin(ny)$, then

$$
\sum_{n=1}^{\infty} nB_n\sin(ny) = \sum_{n=0}^{\infty} A_n\sin(ny) \Rightarrow nB_n = A_n
$$

then we plug in the other boundary condition

$$
u(\pi,y) = \sum_{n=1}^{\infty} nB_n\cosh(n\pi)\sin(ny) + B_n\sinh(n\pi)\sin(ny) = 1
$$

Then treating factoring out the sin and treating the rest as one big constant gives us

$$\sum_{n=1}^{\infty} [nB_n \cosh(n\pi) + B_n \sinh(n\pi)] \sin(ny) = 1$$

$$\Rightarrow nB_n \cosh(n\pi) + B_n \sinh(n\pi) = \frac{2}{\pi} \int_0^{\pi} \sin(ny)dy = -\frac{2}{\pi}\frac{1}{n} \cos(ny)\Big|_0^{\pi} = \frac{2}{n\pi} [1 - (-1)^n]$$

$$\Rightarrow B_n = \frac{\frac{2}{n\pi} [1 - (-1)^n]}{n \cosh(n\pi) + \sinh(n\pi)}$$

Then our full solution is

$$u(x, y) = \sum_{n=1}^{\infty} \frac{\frac{2}{n\pi} [1 - (-1)^n]}{n \cosh(n\pi) + \sinh(n\pi)} [n \cosh(nx) \sin(ny) + \sinh(nx) \sin(ny)] . \tag{12}$$