13.3 Wave Equation Examples

Consider the Wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

with the following boundary and initial conditions

Ex: $u(0,t) = u(L,t) = 0$; $u(x,0) = f(x), \partial_t u(x,0) = g(x)$.

Solution: We make the Ansatz: $u(x,t) = T(t)X(x)$ and plug it into the PDE.

$$u_{tt} = T''(t)X(x), u_{xx} = T(t)X''(x) \Rightarrow T''X = c^2TX'' \Rightarrow \frac{T''}{c^2T} = \frac{X''}{X}$$

Just as in the heat equation we need to identify our Sturm–Liouville problem. We have a LHS and RHS that are functions of different variables, yet they are equal, so they must equal a constant, say $-\lambda^2$. Then we have

$$\frac{T''}{c^2T} = \frac{X''}{X} = -\lambda^2 \tag{2}$$

Notice that unlike the heat equation, if $\lambda = 0$, we would get $T'' = 0$ which would give us a linear function in t for T, but we know that this is unphysical because if a string is plucked it should be oscillatory in t. So $\lambda \neq 0$. Then we can go straight to the sin and cos case.

$$\frac{T''}{c^2T} = -\lambda^2 \Rightarrow T'' + c^2\lambda^2 T = 0 \Rightarrow T = C_1\cos(c\lambda t) + C_2\sin(c\lambda t).$$

And for the X equation we have our usual Sturm–Liouville problem.

$$\frac{X''}{X} = -\lambda^2 \Rightarrow X'' + \lambda^2 X = 0 \Rightarrow X = D_1\cos\lambda x + D_2\sin\lambda x$$

Now we plug in the boundary conditions

$$X(0) = D_1 = 0; \qquad X(L) = D_2\sin\lambda L = 0 \Rightarrow \lambda = \left(\frac{n\pi}{L}\right) \Rightarrow X = D_2\sin\left(\frac{n\pi x}{L}\right)$$

Then the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L} + B_n\sin\frac{n\pi x}{L}\sin\frac{n\pi ct}{L} \tag{3}$$

Now lets plug in the first initial condition,

$$u(x,0) = \sum_{n=1}^{\infty} A_n\sin\frac{n\pi x}{L} = f(x) \Rightarrow A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx.$$

And the second initial condition gives us

$$\begin{aligned} u_t(x,0) &= \sum_{n=1}^{\infty} -\frac{n\pi c}{L}A_n\sin\frac{n\pi x}{L}\sin\frac{n\pi ct}{L}\bigg|_{t=0} + \frac{n\pi c}{L}B_n\sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}\bigg|_{t=0} \\ &= \sum_{n=1}^{\infty}\frac{n\pi c}{L}B_n\sin\frac{n\pi x}{L} = g(x) \Rightarrow \frac{n\pi c}{L}B_n = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}dx. \end{aligned}$$

Then this gives us the full solution

$$u(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx + \frac{L}{n\pi c}\sin\frac{n\pi x}{L}\sin\frac{n\pi ct}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}dx \tag{4}$$

5) $u(0,t) = u(1,t) = 0$; $u(x,0) = x(1-x)$, $\partial_t u(x,0) = g(x)$.

Solution: Above we had the following general solutions for T and X

$$T = C_1 \cos(c\lambda t) + C_2 \sin(c\lambda t) \tag{5}$$

$$X = D_1 \cos \lambda x + D_2 \sin \lambda x \tag{6}$$

and plugging in our boundary conditions gives us

$$X(0) = D_1 = 0; \qquad X(1) = D_2 \sin \lambda = 0 \Rightarrow \lambda = n\pi \Rightarrow X = D_2 \cos n\pi x$$

Then our general solution is

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) \cos(n\pi ct) + B_n \sin(n\pi x) \sin(n\pi ct) \tag{7}$$

Plugging in the first initial condition gives us

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) = x(1-x) \Rightarrow A_n = 2\int_0^1 x(1-x)\sin(n\pi x)dx = \frac{4}{n^3\pi^3}\left((-1)^{n+1}+1\right)$$

And for the second initial condition

$$u_t(x,0) = \sum_{n=1}^{\infty} -(n\pi c)A_n \sin(n\pi x)\sin(n\pi ct)\Big|_{t=0} + (n\pi c)B_n \sin(n\pi c)\cos(n\pi ct)\Big|_{t=0}$$

$$= \sum_{n=1}^{\infty}(n\pi c)B_n \sin(n\pi x) = x(1-x) \Rightarrow (n\pi c)B_n = 2\int_0^1 x(1-x)\sin(n\pi x)dx = \frac{4}{n^3\pi^3}\left((-1)^{n+1}+1\right)$$

Then the full solution is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4}{n^3\pi^3}\left((-1)^{n+1}+1\right)\sin(n\pi x)\left[\cos(n\pi ct) + \frac{4}{n\pi c}\sin(n\pi ct)\right]$$