1. Heat Equation

$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2} \quad (1)$$

- Plug in separation of variables Ansatz into the equation to get the ODEs

$$T' = -K\lambda^2 T; \qquad X'' + \lambda^2 X = 0. \quad (2)$$

- The T solution is easiest: $T = e^{-K\lambda^2 t}$.
- The X equation gives us a Sturm–Liouville problem:
  - Solve for $\lambda = 0$: $X = c_1 x + c_2$. Plug in BCs to solve for constants.
  - Solve for $\lambda \neq 0$: $X = A\cos\lambda x + B\sin\lambda x$. Plug in BCs to solve for one constant and the eigenvalue λ^2.
- Write the general solution $u(x,t) = TX$ as a Fourier series.
- Plug in the **one** initial condition to find the other constants for the complete solution.

1.1. Nonhomogeneous BCs.

- Find the equilibrium solution $u_*(x)$ by solving $u_{xx} = 0$.
- Plug in the nonhomogeneous BCs to u_* to solve for the constants.
- Make the change of variables $v(x,t) = u(x,t) - u_*(x)$.
- Plug back into the original nonhomogeneous PDE to get

$$\frac{\partial v}{\partial t} = K\frac{\partial^2 v}{\partial x^2}; \qquad v(0,t) = v(L,t) = 0; \qquad v(x,0) = u(x,0) - u_*(x). \quad (3)$$

2. Wave Equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \quad (4)$$

- Plug in separation of variables Ansatz into the equation to get the ODEs

$$T'' + c^2\lambda^2 T = 0; \qquad X'' + \lambda^2 X = 0 \quad (5)$$

- Because T must be sinusoidal, $\lambda \neq 0$, so the ODEs give the solutions

$$T = C_1\cos c\lambda t + C_2\sin c\lambda t; \qquad X = D_1\cos\lambda x + D_2\sin\lambda x \quad (6)$$

- Plug the BCs into X to get one constant and the eigenvalue λ^2.
- Write the general solution $u(x,t) = TX$ as a Fourier series.
- Plug in the **two** initial conditions to find the other constants for the complete solution.

3. Laplace Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \quad (7)$$

- Plug in the separation Ansatz into the equation to get $\frac{X''}{X} = -\frac{Y''}{Y}$.
- Determine which direction gives us a Sturm–Liouville problem; i.e., which direction has homogeneous BCs. If it is the x direction, let the equation above equal $-\lambda^2$. If it is the y direction, let the equation equal λ^2.
- The Sturm–Liouville problem will give you **sine** and **cosine** solutions and the other ODE will give you **sinh** and **cosh** solutions.
- Solve the Sturm–Liouville problem for $\lambda = 0$ and $\lambda \neq 0$.
- Plug in the homogeneous BCs into the Sturm–Liouville solution to find the constants and eigenvalue.
- Write the general solution $u(x,y) = XY$ as a Fourier series.
- Plug in the nonhomogeneous boundary conditions to find the other constants for the complete solution.