

LECTURE SEVEN: THE DOUBLE PENDULUM

Consider a double pendulum such as the one in Figure 1, where the first pendulum has a massless rod of length L_1 and mass of m_1 , and the second pendulum has a massless rod of length L_2 and mass of m_2 . Let φ_1 and φ_2 be the angles the respective rods make with respect to the vertical axis.

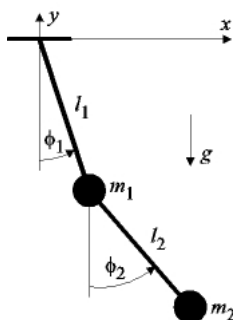


FIGURE 1

The vertical distance of the first mass from equilibrium will be $L_1(1 - \cos \varphi_1)$, hence the potential energy of the first mass will be

$$U_1 = m_1 g L_1 (1 - \cos \varphi_1). \quad (1)$$

The vertical distance of the second mass from its equilibrium with respect to the first mass will be $L_2(1 - \cos \varphi_2)$, but the position of the first mass is $L_1(1 - \cos \varphi_1)$ hence the potential energy of the second mass will be

$$U_2 = m_2 g [L_2(1 - \cos \varphi_2) + L_1(1 - \cos \varphi_1)]. \quad (2)$$

Therefore, the total potential energy will be

$$U = U_1 + U_2 = (m_1 + m_2) g L_1 (1 - \cos \varphi_1) + m_2 g L_2 (1 - \cos \varphi_2). \quad (3)$$

Now for the kinetic energy, the tangential velocity of the first mass will be $L_1 \dot{\varphi}_1$, hence the kinetic energy of the first mass will be

$$T_1 = \frac{1}{2} m_1 L_1^2 \dot{\varphi}_1^2. \quad (4)$$

The velocity of the second mass with respect to the first mass in the tangential direction will be $L_1 \dot{\varphi}_2$, but we must also consider the velocity contribution from the first mass which occurs at the angle $\varphi_2 - \varphi_1$, so our kinetic energy becomes,

$$T_2 = \frac{1}{2} m_2 [L_1^2 \dot{\varphi}_1^2 + 2L_1 L_2 \dot{\varphi}_1 \dot{\varphi}_2 \cos(\varphi_1 - \varphi_2) + L_2^2 \dot{\varphi}_2^2]. \quad (5)$$

Therefore, the total kinetic energy will be

$$T = T_1 + T_2 = \frac{1}{2}(m_1 + m_2)L_1^2\dot{\varphi}_1^2 + m_2L_1L_2\dot{\varphi}_1\dot{\varphi}_2 \cos(\varphi_1 - \varphi_2) + \frac{1}{2}m_2L_2^2\dot{\varphi}_2^2. \quad (6)$$

Most texts employ a small angle approximation at this point in order to simplify the modeling, and to end up with an analytic solution. We, on the other hand, make no such approximation, which will necessitate the use of numerics.

Now, the Lagrangian is

$$\mathcal{L} = T - U = \frac{1}{2}(m_1 + m_2)L_1^2\dot{\varphi}_1^2 + m_2L_1L_2\dot{\varphi}_1\dot{\varphi}_2 \cos(\varphi_1 - \varphi_2) + \frac{1}{2}m_2L_2^2\dot{\varphi}_2^2 - (m_1 + m_2)gL_1(1 - \cos \varphi_1) - m_2gL_2(1 - \cos \varphi_2). \quad (7)$$

So,

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_1} &= (m_1 + m_2)L_1^2\dot{\varphi}_1 + m_2L_1L_2\dot{\varphi}_2 \cos(\varphi_1 - \varphi_2) \\ \Rightarrow \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_1} &= (m_1 + m_2)L_1^2\ddot{\varphi}_1 + m_2L_1L_2\ddot{\varphi}_2 \cos(\varphi_1 - \varphi_2) \\ &\quad - (\dot{\varphi}_1 - \dot{\varphi}_2)m_2L_1L_2\dot{\varphi}_2 \sin(\varphi_1 - \varphi_2). \end{aligned} \quad (8)$$

and

$$\frac{\partial \mathcal{L}}{\partial \varphi_1} = -m_2L_1L_2\dot{\varphi}_1\dot{\varphi}_2 \sin(\varphi_1 - \varphi_2) - (m_1 + m_2)gL_1 \sin \varphi_1. \quad (9)$$

Further,

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_2} &= m_2L_2^2\dot{\varphi}_2 + m_2L_1L_2\dot{\varphi}_1 \cos(\varphi_1 - \varphi_2) \\ \Rightarrow \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_2} &= m_2L_2^2\ddot{\varphi}_2 + m_2L_1L_2\ddot{\varphi}_1 \cos(\varphi_1 - \varphi_2) \\ &\quad - (\dot{\varphi}_1 - \dot{\varphi}_2)m_2L_1L_2\dot{\varphi}_1 \sin(\varphi_1 - \varphi_2). \end{aligned} \quad (10)$$

and

$$\frac{\partial \mathcal{L}}{\partial \varphi_2} = m_2L_1L_2\dot{\varphi}_1\dot{\varphi}_2 \sin(\varphi_1 - \varphi_2) - m_2gL_2 \sin \varphi_2. \quad (11)$$

Now, the equation for $\ddot{\varphi}_1$ is

$$\begin{aligned} \ddot{\varphi}_1 &= \frac{1}{(m_1 + m_2)L_2 - m_2L_1 \cos^2(\varphi_1 - \varphi_2)} \{ -m_2L_2 \cos(\varphi_1 - \varphi_2) [(\dot{\varphi}_1 - \dot{\varphi}_2)L_1\dot{\varphi}_1 \sin(\varphi_1 - \varphi_2) \\ &\quad + L_1\dot{\varphi}_1\dot{\varphi}_2 \sin(\varphi_1 - \varphi_2) + g \sin \varphi_2] + (\dot{\varphi}_1 - \dot{\varphi}_2)m_2L_2\dot{\varphi}_2 \sin(\varphi_1 - \varphi_2) \\ &\quad - m_2L_2\dot{\varphi}_1\dot{\varphi}_2 \sin(\varphi_1 - \varphi_2) - (m_1 + m_2)g \sin \varphi_1 \} \end{aligned} \quad (12)$$

and the equation for $\ddot{\varphi}_2$ is

$$\ddot{\varphi}_2 = \frac{L_1}{L_2} \left\{ \frac{-\cos(\varphi_1 - \varphi_2)}{(m_1 + m_2)L_2 - m_2L_1 \cos^2(\varphi_1 - \varphi_1)} \left\{ -m_2L_2 \cos(\varphi_1 - \varphi_2) [(\dot{\varphi}_1 - \dot{\varphi}_2)L_1\dot{\varphi}_1 \sin(\varphi_1 - \varphi_2) \right. \right. \\ \left. \left. + L_1\dot{\varphi}_1\dot{\varphi}_2 \sin(\varphi_1 - \varphi_2) + g \sin \varphi_2 \right\} + (\dot{\varphi}_1 - \dot{\varphi}_2)m_2L_2\dot{\varphi}_2 \sin(\varphi_1 - \varphi_2) \right. \\ \left. - m_2L_2\dot{\varphi}_1\dot{\varphi}_2 \sin(\varphi_1 - \varphi_2) - (m_1 + m_2)g \sin \varphi_1 \right\} + (\dot{\varphi}_1 - \dot{\varphi}_2)L_1\dot{\varphi}_1 \sin(\varphi_1 - \varphi_2) \\ \left. + L_1\dot{\varphi}_1\dot{\varphi}_2 \sin(\varphi_1 - \varphi_2) + g \sin \varphi_2 \right\} \quad (13)$$

Suppose we want to make our lives harder and modeled a double pendulum with arms of uniform nontrivial mass. We need to find the center of mass for each arm in relation to the angles θ_1 and θ_2 ,

$$x_1 = \frac{L_1}{2} \sin \theta_1,$$

$$x_2 = L_1 \sin \theta_1 + \frac{L_2}{2} \sin \theta_2,$$

$$y_1 = -\frac{L_2}{2} \cos \theta_1,$$

$$y_2 = -L_1 \cos \theta_1 + \frac{L_2}{2} \cos \theta_2.$$

We also have to include rotational kinetic energy,

$$\frac{1}{2} [I_1\dot{\theta}_1^2 + I_2\dot{\theta}_2^2] = \frac{1}{24} [m_1L_1^2\dot{\theta}_1^2 + m_2L_2^2\dot{\theta}_2^2],$$

the the Lagrangian is,

$$\mathcal{L} = \frac{1}{2} [m_1 (\dot{x}_1^2 + \dot{y}_1^2) + m_2 (\dot{x}_2^2 + \dot{y}_2^2)] + \frac{1}{24} [m_1L_1^2\dot{\theta}_1^2 + m_2L_2^2\dot{\theta}_2^2] - (m_1gy_1 + m_2gy_2), \quad (14)$$

where

$$\dot{x}_1 = \frac{L_1}{2} \dot{\theta}_1 \cos \theta_1,$$

$$\dot{x}_2 = L_1 \dot{\theta}_1 \cos \theta_1 + \frac{L_2}{2} \dot{\theta}_2 \cos \theta_2,$$

$$\dot{y}_1 = \frac{L_2}{2} \dot{\theta}_1 \sin \theta_1,$$

$$\dot{y}_2 = L_2 \dot{\theta}_1 \sin \theta_1 - \frac{L_2}{2} \dot{\theta}_2 \sin \theta_2.$$

After this point the algebra becomes quite tedious, but if anyone is adventurous enough they can crank it out on Mathematica.