11.7 EXTREMA

Ex: Find the local maximum and minimum values and saddle points of $f(x,y) = x^4 + y^4 - 4xy + 1$.

Solution: First we take the derivatives and find the critical points.

$$f_x = 4x^3 - 4y = 0, \quad f_y = 4y^3 - 4x = 0 \Rightarrow (x_*, y_*) = (0,0),\ (1,1),\ (-1,-1)$$

Next we apply the second derivative test.

$$f_{xx} = 12x^2,\ f_{x,y} = f_{yx} = -4,\ f_{yy} = 12y^2 \Rightarrow H = 144x^2y^2 - 16.$$

Then $H(0,0) = -16 < 0$, so this is a saddle point, $H(1,1) = 128 > 0$ and $f_{xx}(1,1) = 12 > 0$ so this is a minimum, and $H(-1,-1) = 128 > 0$ and $f_{xx}(-1,-1) = 12 > 0$ so this is a minimum as well.

Ex: Find the shortest distance from point $(1,0,-2)$ to the plane $x + 2y + z = 4$.

Solution: Recall that the distance from any arbitrary point to the point $(1,0,-1)$ is $d = \sqrt{(x-1)^2 + y^2 + (z+2)^2}$, but we are looking at distances from the plane $z = 4 - x - 2y$, so $d = \sqrt{(x-1)^2 + y^2 + (6-x-2y)^2}$. We can minimize d by minimizing the simpler expression

$$d^2 = f(x,y) = (x-1)^2 + y^2 + (6-x-2y)^2.$$

Then

$$f_x = 4x + 4y - 14 = 0,\ f_y = 4x + 10y - 24 = 0 \Rightarrow (x_*, y_*) = \left(\frac{11}{6}, \frac{5}{3}\right).$$

Next

$$f_{xx}\left(\frac{11}{6}, \frac{5}{3}\right) = 4 > 0,\ f_{xy}\left(\frac{11}{6}, \frac{5}{3}\right) = f_{yx}\left(\frac{11}{6}, \frac{5}{3}\right) = 4\ f_{yy}\left(\frac{11}{6}, \frac{5}{3}\right) = 10,$$

then $H(11/6, 5/3) = 24 > 0$, so the critical point is a minima (as we expected it would be). Then

$$d\left(\frac{11}{6}, \frac{5}{3}\right) = \sqrt{\left(\frac{5}{6}\right)^2 + \left(\frac{5}{3}\right)^2 + \left(\frac{5}{6}\right)^2} = \frac{5}{\sqrt{6}}. \tag{1}$$

Theorem 1 (Extreme value)**.** *If f is continuous on a closed, bounded set $D \subset \mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_2)$ and an absolute minimum value $f(x_2, y_2)$ at points (x_1, y_1) and (x_2, y_2) in D.*

Lets see what this means in English.

How to find absolute maxima and minima of continuous functions

(1) Find the values of f at the critical points of D.
(2) Find the extreme values of f on the boundary of D.
(3) The largest (smallest) values from 1) and 2) is the absolute maximum (minimum).

Ex: Find the absolute maximum and minimum values of $f(x,y) = x^2 - 2xy + 2y$ on rectangle $D = \{(x,y) | 0 \le x \le 3,\ 0 \le y \le 2\}$.

Solution:

Step 1: $f_x = 2x - 2y = 0$, $f_y = -2x + 2 = 0$, then the critical point is $(x_*, y_*) = (1,1)$ and $f(1,1) = 1$.

Step 2: We have to now test each side of our rectangular domain (if you had a circular domain you would have to convert the function and domain into polar coordinates and test the function at the radius.) We will notice that the function $f(x,y)$ turns into a single variable function, so we need to look for max and min just as we did in Calc I for single variable functions: find the critical points and end points and evaluate.

$y=0$: $f(x,0)=x^2$; $0\le x\le 3$. Critical points: $f(0,0)=0$. End points: $f(3,0)=9$, $f(0,0)=0$. Notice that the critical point here is also an end point.

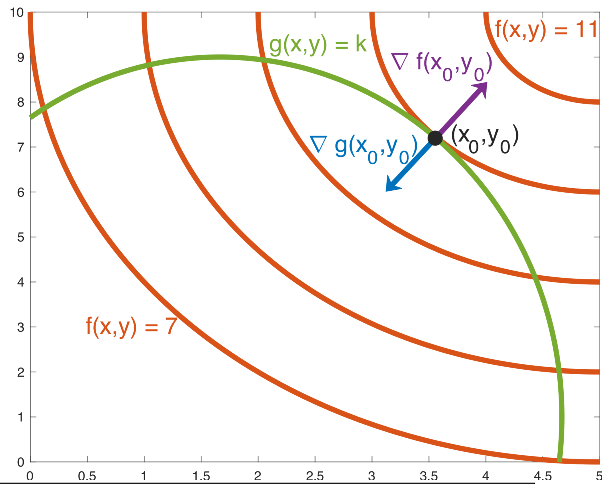
$x=3$: $f(3,y)=9-4y$; $0\le y\le 2$. Critical points: None. End points: $f(3,0)=9$, $f(3,2)=1$. There are no critical points here because the function is linear.

$y=2$: $f(x,2)=x^2-4x+4=(x-2)^2$; $0\le x\le 3$. Critical points: $f(2,2)=0$. End points: $f(0,2)=4$, $f(3,2)=1$. Here we have one critical point in the middle. You can see this by differentiating and setting it to zero.

$x=0$: $f(0,y)=2y$; $0\le y\le 2$. Critical points: None. End points: $f(0,0)=0$, $f(0,2)=4$. Again, no critical points because it's a linear function.

Step 3: Absolute maximum: $\boxed{f(3,0)=9}$, Absolute minimum: $\boxed{f(0,0)=f(2,2)=0}$.

11.8 Lagrange multipliers



We know the fastest route is in the direction of the gradient; i.e., straight up the mountain. In real life roads what is the route up a mountain? Switchbacks; that is, the fastest route along a particular level curve. Lets look at an easy example. Suppose we want to maximize $f(x,y)$ subject to the constraint $g(x,y)=k$. Notice that the maximum is the value c where the normal is in the same direction for g and f. What is the value of the maximum c on the plot? 10. This means that f and g will have the same gradient at (x_0,y_0) up to a scalar multiple; i.e., $\nabla f(x_0,y_0)=\lambda\nabla g(x_0,y_0)$. The scalar multiple λ is called the Lagrange multiplier.

(1) Find all values of x, y, z, and λ such that

$$\nabla f(x,y,z)=\lambda\nabla g(x,y,z) \tag{2}$$

and

$$g(x,y,z)=k. \tag{3}$$

(2) Evaluate f at all those values. Then the smallest value of f is the minimum and the largest is the maximum.

Ex: Difficulty: Moderate A rectangular box (without the lid) is to be made of 12 mm^2 of cardboard. Find the maximum volume of such a box.

Solution: The volume is $V=xyz$ and the surface area is $A=2xz+2yz+xy=12$.

Step 1: First we calculate the two gradients,

$$\nabla V=\langle yz,xz,xy\rangle \quad\text{and}\quad \nabla A=\langle 2z+y,2z+x,2x+2y\rangle$$

and since $\nabla V=\lambda\nabla A$ we get the following system of equations

$$\begin{aligned}yz&=\lambda(2z+y),\\ xz&=\lambda(2z+x),\\ xy&=\lambda(2x+2y),\\ 2xz+2yz+xy&=12.\end{aligned}$$

Step 2: Now we have to solve the system of equations. Notice that if we multiply the first three equations by x, y, and z respectively we get equivalent left hand sides

$$\begin{aligned} xyz &= \lambda(2xz + xy), \\ xyz &= \lambda(2yz + xy), \\ xyz &= \lambda(2xz + 2yz), \\ \Rightarrow 2xz + xy &= 2yz + xy = 2xz + 2yz. \end{aligned}$$

From the first equality, if $z \neq 0$, we notice that

$$2xz + xy = 2yz + xy \Rightarrow 2xz = 2yz \Rightarrow \boxed{x = y}$$

and from the second equality, again if $x \neq 0$, we get

$$2yz + xy = 2xz + 2yz \Rightarrow xy = 2xz \Rightarrow \boxed{y = 2z}.$$

Notice that it is not a problem keeping z and x away from zero since our physical problem would collapse if the former were true.

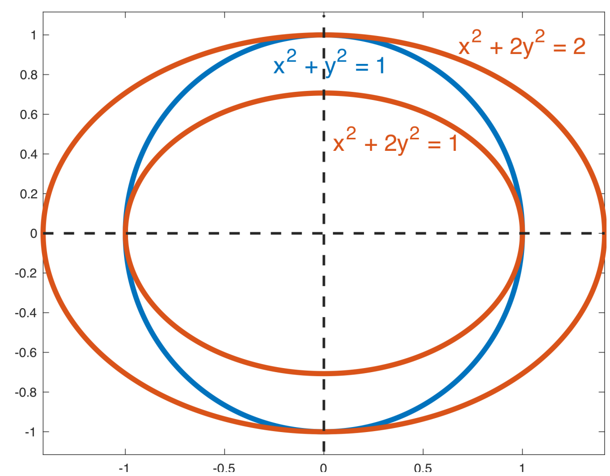
Finally, we write x and y in terms of z and plug it into the last equation

$$2xz + 2yz + xy = 12z^2 = 12 \Rightarrow \boxed{z = 1, x = y = 2}.$$

We could easily find λ, but it is not essential for this particular problem since we were able to solve it without using λ.

Then our maximum volume is $V = 4$ m^3.

Ex: Difficulty: Easy Find the extreme values of $f(x, y) = x^2 + 2y^2$ on circle $x^2 + y^2 = 1$.



Solution: Here the problem is set up for us, so we jump right in

Step 1: $\nabla f = \lambda \nabla g \Rightarrow \langle 2x, 4y \rangle = \lambda \langle 2\lambda x, 2\lambda y \rangle$, then our system of equation is

$$\begin{aligned} 2x &= 2\lambda x, \\ 4y &= 2\lambda y, \\ x^2 + y^2 &= 1 \end{aligned}$$

Step 2: From the first equation, either $x = 0$ or $x \neq 0 \Rightarrow \lambda = 1$, so if $x \neq 0$, we plug $\boxed{\lambda = 1}$ into the second equation to get $\boxed{y = 0}$. If we plug $y = 0$ into the third equation we get $\boxed{x = \pm 1}$, and $f(\pm 1, 0) = 1$. On the other hand, if $x = 0$, $y = \pm 1$ when we plug x into the third equation, and $f(0, \pm 1) = 2$.

Then the maximum is $f(0, \pm 1) = 2$, and the minimum is $f(\pm 1, 0) = 1$.

Ex: Find the extreme values of $f(x, y) = x^2 + 2y^2$ on disk $x^2 + y^2 \leq 1$.

Solution: From the previous example the maximum must be $f(0, \pm 1) = 2$, but since we are no long restricted to the circle, we can go all the way to the origin, so the minimum is $f(0, 0) = 0$.

Ex: Difficulty: Hard Find points on the sphere $x^2 + y^2 + z^2 = 14$ that are closest to and farthest from point $(3, 1, -1)$.

Solution: The function we want to maximize/minimize is the distance from an arbitrary point to $(3, 1, -1)$ and our constraint is the sphere. The distance is $d = \sqrt{(x-3)^2 + (y-1)^2 + (z+1)^2}$, but just as before it is easier to work with the square and it gives us the same results, so

$$d^2 = f(x, y, z) = (x-3)^2 + (y-1)^2 + (z+1)^2$$

and the constraint is

$$g(x, y, z) = x^2 + y^2 + z^2 = 4.$$

Step 1: Just as before we take the gradients and set them equal with a Lagrange multiplier

$$\nabla f = \lambda \nabla g \Rightarrow \langle 2(x-3), 2(y-1), 2(z+1) \rangle = \lambda \langle 2\lambda x, 2\lambda y, 2\lambda z \rangle$$

then the system of equations is

$$\begin{aligned} 2(x-3) &= 2\lambda x \\ 2(y-1) &= 2\lambda y \\ 2(z+1) &= 2\lambda z \\ x^2 + y^2 + z^2 &= 4. \end{aligned}$$

Step 2: Solving the first three equations gives us

$$x = \frac{3}{1-\lambda}, \quad y = \frac{1}{1-\lambda}, \quad z = -\frac{1}{1-\lambda}$$

and plugging this into the last equation gives us

$$\frac{3^2}{(1-\lambda)^2} + \frac{1}{(1-\lambda)^2} + \frac{(-1)^2}{(1-\lambda)^2} = 4 \Rightarrow (1-\lambda)^2 = \frac{11}{4} \Rightarrow \boxed{\lambda = 1 \pm \frac{\sqrt{11}}{2}},$$

which solves x, y, and z. Then the closest point is

$$\boxed{f\left(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, -\frac{2}{\sqrt{11}}\right)}$$

and the farthest point is

$$\boxed{f\left(-\frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)}$$