Math 2450 Rahman Exam II Sample Problems

(1) Compute the following limits or show it does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{xy} \tag{1}$$

$$\lim_{(x,y)\to(1,1)} \frac{(x-1)^2-(y-1)^2}{x-y} \tag{2}$$

$$\lim_{(x,y)\to(0,0)} \cos xy \tag{3}$$

$$\lim_{(x,y)\to(1,1)} \frac{x^2+y^2}{xy} \tag{4}$$

$$\lim_{(x,y)\to(1,0)} \left(\tan^{-1}\frac{x}{y}\right)^2 \tag{5}$$

Solution:

(a) Take $y = x$, then the limit is 2, then take $y = x^2$, then the limit is ∞, so the limit DNE because we found two limits that were different.

(b) Notice that $x - y = (x-1) - (y-1)$, then the denominator is one factor of the numerator,

$$\lim_{(x,y)\to(1,1)} \frac{(x-1)^2-(y-1)^2}{x-y} = \lim_{(x,y)\to(1,1)} \frac{(x-1)^2-(y-1)^2}{(x-1)-(y-1)} = \lim_{(x,y)\to(1,1)} (x-1)+(y-1) = 0.$$

(c) 1.

(d) 2.

(e) Limit DNE. Take $y = mx$, then you get $(\tan^{-1}(1/m))^2$ which is going to be different for different values of m.
Interesting side note: if it was x/y^2 instead, then the limit would exists since $x/y^2 \to \infty$ and $\tan^{-1}(\infty) = \pi/2$.

(2) Find the equation of the tangent plane to the graph of the function $f(x,y) = (x+2y)\cos(3xy)$ at point $(0,1,2)$.

Solution: If $f(x,y) = z$, let $g(x,y,z) = (x+2y)\cos(3xy) - z = 0$, then

$$\nabla g = \langle \cos(3xy) - 3y(x+2y)\sin(3xy), 2\cos(3xy) - 3x(x+2y)\sin(3xy), -1\rangle \Rightarrow \nabla g(0,1,2) = \langle 1,2,-1\rangle$$

Then the equation for the plane is

$$3(x-0) + 6(y-1) - 1(z-2) = 0.$$

(3) The total resistance R of two resistors with resistance R_1 and R_2 connected in parallel is given by the following well-known formula:

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

Suppose that R_1 and R_2 are measured to be 300 and 600 ohms, respectively, with an error of ± 15 ohms in each measurement. Use differentials to estimate the maximum error in ohms in the calculated value of R.

Solution: We implicitly differentiate to get

$$\frac{-1}{R^2}dR = \frac{-1}{R_1^2}dR_1 + \frac{-1}{R_2^2}dR_2 \Rightarrow dR = \left(\frac{1}{R_1^2}dR_1 + \frac{1}{R_2^2}dR_2\right)R^2 = \left(\frac{1}{R_1^2}dR_1 + \frac{1}{R_2^2}dR_2\right)\left(\frac{1}{R_1}+\frac{1}{R_2}\right)^{-2}$$

and plugging in the resistance gives us

$$dR = \left(\frac{\pm 15}{300^2} + \frac{\pm 15}{600^2}\right)\left(\frac{1}{300}+\frac{1}{600}\right)^{-2}$$

(4) Consider the surface described implicitly by the equation

$$2yx^3 + \frac{z^2}{x} + xy\ln z = 3$$

(a) Find the equation of the tangent plane to the surface at point $(1, 1, 1)$

Solution: The normal vector will be of the form $\langle \partial z/\partial x, \partial z/\partial y, -1\rangle$. The implicit derivatives will be

$$\begin{aligned} 6yx^2 - \frac{z^2}{x^2} + \frac{2z}{x}\frac{\partial z}{\partial x} + y\ln z + \frac{xy}{z}\frac{\partial z}{\partial x} &= 0 \\ 2x^3 + \frac{2z}{x}\frac{\partial z}{\partial y} + x\ln z + \frac{xy}{z}\frac{\partial z}{\partial y} &= 0 \end{aligned}$$

We plug in the points before solving for $\partial z/\partial x$ and $\partial z/\partial y$ because it makes the calculation much easier

$$\begin{aligned} 6 - 1 + 2\frac{\partial z}{\partial x} + 0 + \frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} &= -\frac{5}{3} \\ 2 + 2\frac{\partial z}{\partial y} + 0 + \frac{\partial z}{\partial y} = 0 \Rightarrow \frac{\partial z}{\partial y} &= -\frac{2}{3} \end{aligned}$$

Then the equation for the tangent plane is

$$\frac{5}{3}(x-1) + \frac{2}{3}(y-1) + (z-1) = 0.$$

(b) Using the linear approximation evaluate the approximate value of z on the surface when $x = 1.01$ and $y = 0.98$.

Solution: Plugging in the values into the equation for the tangent plane gives us

$$z \approx 1 - \frac{5}{3}(1.01 - 1) - \frac{2}{3}(0.98 - 1)$$

(5) Determine the local extrema locations (critical points) for

$$z = xy^2 + \frac{1}{2}x^2 + y^2 + 10$$

Solution: $f_x = y^2 + x = 0$, $f_y = 2xy + 2y = 0$, then if $y = 0$, $x = 0$ if $y \neq 0$, $2x + 2 = 0 \Rightarrow x = -1 \Rightarrow y = \pm 1$, so the critical points are $(0, 0)$, $(-1, 1)$, and $(-1, -1)$.

(6) Find and classify (max, min, saddle, inconclusive) the critical points of

$$z = 2(x+1)^2 + 3(y-2)^2 + 6(y-2)$$

Solution: $f_x = 4(x + 1) = 0$ and $f_y = 6(y - 2) + 6 = 0$, then $x = -1$ and $y = 1$, so the critical point is $(-1, 1)$. The second derivatives are $f_{xx} = 4$, $f_{yy} = 6$, and $f_{xy} = f_{yx} = 0$, so the Hessian is

$$H(-1, 1) = \begin{vmatrix} 4 & 0 \\ 0 & 6 \end{vmatrix} = 24 > 0$$

Since $H(-1, 1) > 0$ and $f_{xx}(-1, 1) > 0$, $f(-1, 1)$ is a minima.

(7) Determine the following derivatives, using chain rule, for $w = xe^z + zy$

(a) dw/dt at $t = 1$, where $x = 1/t$, $y = t^3$, and $z = t - 1$

Solution:

$$\frac{dw}{dt} = e^z\frac{dx}{dt} + xe^z\frac{dz}{dt} + y\frac{dz}{dt} + z\frac{dy}{dt} = e^z\frac{-1}{t^2} + xe^z + y + z(3t^2).$$

plugging in our point gives us $x = 1$, $y = 1$, and $z = 0$, then

$$\frac{dw}{dt} = -1 + 1 + 1 + 0 = 1.$$

(b) $\partial w/\partial v$ at $u = 1$ and $v = 1$, where $x = u^2 + v$, $y = uv^2$, and $z = v^2 - u^2$.

Solution:

$$\frac{\partial w}{\partial v} = e^z\frac{\partial x}{\partial v} + xe^z\frac{\partial z}{\partial v} + y\frac{\partial z}{\partial v} + z\frac{\partial y}{\partial v} = e^z + xe^z(2v) + y(2v) + z(2uv).$$

plugging in our points give us $x = 2$, $y = 1$, and $z = 0$, then

$$\frac{\partial w}{\partial v} = 1 + 4 + 2 + 0 = 7.$$

(8) For the surface given by the equation $x^2yz + x + yz^3 = 7$, determine the following at point $(2, 1, 1)$

(a) The equation of the plane tangent to the surface

Solution:

$$\nabla f = \langle 2xyz + 1, x^2z + z^3, x^2y + 3yz^2\rangle$$

and after plugging in the points we get

$$\nabla f(2, 1, 1) = \langle 5, 5, 7\rangle \Rightarrow 5(x - 2) + 5(y - 1) + 7(z - 1) = 0.$$

(b) write $\partial z/\partial x$ only in terms of x, y, z, and constants (just like we did in class).

Solution: We could use the implicit function theorem, but I don't like remembering formulas, so lets derive it from scratch.

$$\frac{\partial}{\partial x}\left[x^2yz + x + yz^3 = 7\right] \Rightarrow 2xyz + x^2y\frac{\partial z}{\partial x} + 1 + 3yz^2\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = -\frac{2xyz + 1}{x^2y + 3yz^2}$$

which is the same thing you would get from the implicit function theorem: $\partial z/\partial x = -f_x/f_z$.

(9) For the function $f(x, y, z) = x/y + 2xyz$, evaluate the following at point $(1, 1, 0)$

(a) The directional derivative in the direction $\vec{v} = \hat{\mathbf{i}} - 2\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$

Solution: $\vec{u} = \langle 1, -2, -2\rangle/3$ and

$$\nabla f = \left\langle \frac{1}{y} + 2yz, \frac{-x}{y^2} + 2xz, 2xy\right\rangle \Rightarrow \nabla f(1, 1, 0) = \langle 1, -1, 1\rangle$$

then $D_u f(1, 1, 0) = 1/3$.

(b) A unit vector in the direction in which the directional derivative is maximum

Solution:

$$\left.\frac{\nabla f}{\|\nabla f\|}\right|_{(1,1,0)} = \frac{\langle 1, -1, 1\rangle}{\sqrt{3}}.$$

(c) The maximum value of the directional derivative

Solution:

$$\|\nabla f\| = \sqrt{3}.$$

(10) Using Lagrange multipliers, find the point on the plane $x + 2y + 3z = 6$ that is closest to the origin $(0, 0, 0)$.

Solution: We first use the distance function $d = \sqrt{x^2 + y^2 + z^2}$ since the reference point is the origin. Let $d^2 = f(x, y, z) = x^2 + y^2 + z^2$ with constraint $g(x, y, z) = x + 2y + 3z = 6$.

Step 1: Then we take the gradients $\nabla f = \langle 2x, 2y, 2z \rangle$ and $\nabla g = \langle 1, 2, 3 \rangle$ and our system of equations is

$$\begin{aligned} 2x &= \lambda \\ 2y &= 2\lambda \\ 2z &= 3\lambda \\ x + 2y + 3z &= 6 \end{aligned}$$

then $y = 2x$ and $z = 3x$, and plugging this into the last equation gives us $x + 4x + 9x = 14x = 6$, so the point on the plane that is closest to the origin is $\boxed{(3/7, 6/7, 9/7)}$.