6.2 Cylindrical Shells

Another method to do volumes of revolutions is through cylindrical shells. This method is a lot less intuitive, and hence requires more practice. Basically think of infinitesimal cylinders filling up a region. We know the area of the side of the cylinder is $A = 2\pi rh$. So, by summing up these infinitesimal cylinders we get the following formulas for rotation about the y-axis and x-axis respectively,

$$V = \int_a^b 2\pi x h(x) dx \tag{1}$$

$$V = \int_\alpha^\beta 2\pi y h(y) dy \tag{2}$$

The next few problems for rotation about the axes were done in class.

5) $r(x) = x$ and $h(x) = \sqrt{x^2+1}$, so the area is $A = 2\pi x\sqrt{x^2+1}$, then

$$V = \pi \int_0^{\sqrt{3}} 2x\sqrt{x^2+1}dx$$

We solve this via u-sub with $u = x^2 + 1 \Rightarrow du = 2xdx$,

$$V = \pi \int_1^4 u^{1/2}du = \frac{2\pi}{3}u^{3/2}\Big|_1^4 = \frac{16\pi}{3} - \frac{2\pi}{3} = \frac{14\pi}{3}.$$

17) Intersection: $2y - y^2 = 0 \Rightarrow y(y-2) = 0 \Rightarrow y = 0, 2$. $r(y) = y$, $h(y) = 2y - y^2$, then the area is $A = 2\pi\left[2y^2 - y^3\right]$. So, the volume is

$$V = 2\pi \int_0^2 (2y^2 - y^3)dy = 2\pi \left[\frac{2}{3}y^3 - \frac{1}{4}y^4\right]_0^2 = 2\pi\left[\frac{16}{3} - 4\right] = \frac{8\pi}{3}.$$

29) Intersection: $x = x^2 \Rightarrow x^2 - x = x(x-1) = 0 \Rightarrow x = 0, 1$. $r(x) = x$, and $h(x) = x - x^2 \Rightarrow A = 2\pi[x^2 - x^3]$, then

$$V = 2\pi \int_0^1 (x^2 - x^3)dx = 2\pi\left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^1 = \frac{\pi}{6}.$$

What happens if the line of rotation is not one of the axes? Suppose the line of rotation is $x = x_0$, then in general $|x_0 - x|$.

33) Intersection: $y - y^3 = 0 \Rightarrow y(1-y)(1+y) = 0 \Rightarrow y = 0, \pm 1$, $r(y) = 1 - y$ and $h(y) = y - y^3$, so $A = 2\pi(1-y)(y-y^3) = 2\pi[y - y^3 - y^2 + y^4]$. Then the volume is

$$V = 2\pi \int_0^1 (y^4 - y^3 - y^2 + y)dy = 2\pi\left[\frac{1}{5}y^5 - \frac{1}{4}y^4 - \frac{1}{3}y^3 + \frac{1}{2}y^2\right]_0^1 = 2\pi\left[-\frac{1}{20} + \frac{1}{12}\right] = \frac{\pi}{15}.$$

Now lets do a problem that is much easier with cylindrical shells than disks/washers.

47) $r(x) = x$ and $h(x) = e^{-x^2}$, then $A = 2\pi rh = 2\pi x e^{-x^2}$, so the volume is

$$V = \pi \int_0^1 2xe^{-x^2}dx.$$

We solve this via u-sub, $u = x^2 \Rightarrow du = 2xdx$, then

$$V = \pi \int_0^1 e^{-u}du = -\pi e^{-u}\Big|_0^1 = -\pi e^{-1} + \pi = \pi\left(1 - \frac{1}{e}\right).$$

Additional problems (not from the book):

(1) Find the volume of the region bounded by $y = 2x^2 - x^3$ and $y = 0$ revolved around the y-axis.

Solution: Here the radius of each cylinder will be $r = x$ and height will be $h = y = 2x^2 - x^3$. Then

$$V = \int_a^b 2\pi x f(x) dx = \int_0^2 2\pi x(2x^2 - x^3) dx = \frac{16}{5}\pi$$

(2) Find the volume of the region bounded by $y = x$ and $y = x^2$ revolved about the y-axis.

Solution: The radius is $r = x$ and the height is $h = x - x^2$, then

$$V = \int_a^b 2\pi x f(x) dx = \int_0^1 2\pi x(x - x^2) dx = \frac{\pi}{6}.$$

(3) Find the volume of the region bounded by $y = \sqrt{x}$, $x = 0$, and $x = 1$ revolved about the x-axis.

Solution: The radius is $r = y$ and the height is $h = 1 - y^2$, then

$$V = \int_a^b 2\pi y f(y) dy = \int_0^1 2\pi y(1 - y^2) dy = \frac{\pi}{2}.$$

(4) Find the volume of the region bounded by $y = x - x^2$ and $y = 0$ revolved about the line $x = 2$.

Solution: Since our axis of revolution is towards the right, the radius of our cylinders will be $r = 2 - x$ and the height is $h = y = x - x^2$. Hence, our area is $A = 2\pi(2 - x)(x - x^2)$. Then

$$V = \int_0^1 2\pi(2 - x)(x - x^2) dx = \frac{\pi}{2}.$$

6.3 Arc Length

Arc length is just the sum of infinitesimal small pieces of an arc, so we can derive the formula:

$$L = \int_a^b \sqrt{dx^2 + dy^2}. \tag{3}$$

This can then be parameterized in two main ways,

$$L = \int_a^b \sqrt{1 + f'(x)^2} dx \qquad \text{if } f \in C^1([a, b]), \tag{4}$$

$$L = \int_c^d \sqrt{1 + g'(y)^2} dy \qquad \text{if } g \in C^1([c, d]). \tag{5}$$

This means that we use the first formula if $y = f(x)$ has a continuous derivative on $[a, b]$ (the interval between which we are calculating arc length), and we use the second formula if $x = g(y)$ has a continuous derivative on $[c, d]$. If it has a continuous derivative for both we may use either formula.

We did the following problems in class,

2) First we take the derivative then plug it into the integral

$$\frac{dy}{dx} = \frac{3}{2}x^{1/2} \Rightarrow L = \int_0^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_0^4 \sqrt{1 + \frac{9}{4}x dx}.$$

We solve this via u-sub, $u = 1 + 9x/4 \Rightarrow du = (9/4)dx$

$$L = \frac{4}{9}\int_1^{10} \sqrt{u} du = \frac{8}{27}u^{3/2}\Big|_1^1 0 = \frac{8}{27}\left(10^{3/2} - 1\right).$$

4) For this problem lets do some algebra after taking the derivative,

$$\frac{dx}{dy} = \frac{1}{2}y^{1/2} - \frac{1}{2}y^{-1/2} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{1}{4}y - \frac{1}{2} + \frac{1}{4}y^{-1} \Rightarrow 1 + \left(\frac{dx}{dy}\right)^2 = \frac{1}{4}y + \frac{1}{2} + \frac{1}{4}y^{-1} = \left(\frac{1}{2}y^{1/2} + \frac{1}{2}y^{-1/2}\right)^2.$$

Then the length is

$$L = \int_1^9 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_1^9 \left(\frac{1}{2}y^{1/2} + \frac{1}{2}y^{-1/2}\right) dy = \frac{1}{3}y^{3/2} + y^{1/2}\Big|_1^9 = 9 + 3 - \frac{1}{3} - 1 = \frac{32}{3}.$$

Ex: $y^3 = x^2$ from $(0,0)$ to $(8,4)$.

Failed Solution: If we take the usual route, $dy/dx = (2/3)x^{-1/3}$. This is clearly not continuous at $x = 0$, so we need to find another way.

Solution: Let $x = y^{3/2} \Rightarrow dx/dy = (3/2)x^{1/2}$, then

$$L - \int_0^4 \sqrt{1 + \frac{9}{4}y}dy = \frac{8}{27}\left(10^{3/2} - 1\right).$$

Ex: $x = \int_0^y \sqrt{\csc^4 t - 1}dt$ from $y = \pi/4$ to $y = \pi/2$.

Solution: We take the derivative and do some algebra,

$$\frac{dx}{dy} = \sqrt{\csc^4 y - 1} \Rightarrow 1 + \left(\frac{dx}{dy}\right)^2 = \csc^4 y.$$

Then the length is

$$L = \int_{\pi/4}^{\pi/2} \csc^2 y dy = -\cot y\Big|_{\pi/4}^{\pi/2} = 1.$$

6.4 Areas of a surface of revolution

Areas of revolution are similar to volumes of revolution, except now, we integrate over the length of the arc, so for a revolution about the x-axis and y-axis respectively

$$A = 2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}dx \tag{6}$$

$$A = 2\pi \int_c^d g(y)\sqrt{1 + g'(y)^2}dy \tag{7}$$

However, just like arc lengths, we can parameterize this in many ways, but one convenient way to do it is as such

$$A = 2\pi \int_c^d y\sqrt{1 + g'(y)^2}dy \tag{8}$$

$$A = 2\pi \int_a^b x\sqrt{1 + f'(x)^2}dx. \tag{9}$$

We did the following problems in class,

13) We take the derivative and then plug it into the area equation

$$f'(x) = \frac{x^2}{9} \Rightarrow A = 2\pi \int_0^2 \frac{x^3}{3}\sqrt{1 + \frac{x^4}{9}}dx$$

Via u-sub we get $u = 1 + \frac{x^4}{9} \Rightarrow du = \frac{4}{9}x^3 dx$, then

$$A = \frac{\pi}{2}\int_1^{16/9} u^{1/2}du = \frac{\pi}{3}u^{3/2}\Big|_1^{16/9} = \frac{\pi}{3}\left[\frac{4^3}{3^3} - 1\right] = \frac{98\pi}{81}.$$

Ex: $y = x^2$ from $(1,1)$ to $(2,4)$ about the $y-axis$.

Method 1: If we parameterize with x,

$$A = 2\pi \int_1^2 x\sqrt{1+4x^2}dx = \frac{\pi}{4}\int_5^{17} u^{1/2}du = \frac{\pi}{6}u^{3/2}\Big|_5^{17} = \frac{\pi}{6}\left(17^{3/2} - 5^{3/2}\right).$$

Method 2: If we use the usual parameterization,

$$A = 2\pi \int_1^4 \sqrt{y}\sqrt{1+\frac{1}{4y}}dy = 2\pi\int_1^4 \sqrt{y+\frac{1}{4}}dy = 2\pi\int_{5/4}^{17/4} u^{1/2}du = \frac{4\pi}{3}u^{3/2}\Big|_{5/4}^{17/4} = \frac{\pi}{6}\left(17^{3/2} - 5^{3/2}\right).$$

19) We take the derivative and do some algebra

$$\frac{dx}{dy} = -(4-y)^{-1/2} \Rightarrow 1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{1}{4-y}.$$

Then our area equation is

$$A = 2\pi\int_0^{15/4} 2\sqrt{4-y}\sqrt{1+\frac{1}{4-y}}dy = 4\pi\int_0^{15/4}\sqrt{4-y+1}dy = 4\pi\int_0^{15/4}\sqrt{5-y}dy.$$

We solve this via u-sub with $u = 5-y \Rightarrow du = -dy$, then

$$A = -4\pi\int_5^{5/4} u^{1/2}du = -\frac{8}{3}\pi u^{3/2}\Big|_5^{5/4} = -\pi\frac{5\sqrt{5}}{3} + \pi\frac{40\sqrt{5}}{3} = \frac{35\sqrt{5}}{3}\pi.$$

24) For this problem we haven't learned how to solve the integral, but we can set it up

$$\frac{dy}{dx} = -\sin x \Rightarrow A = 2\pi\int_{-\pi/2}^{\pi/2}\cos x\sqrt{1+\sin^2 x}dx.$$

Ex: $x = \frac{y^4}{4} + \frac{1}{8y^2}$ between $1 \le y \le 2$ about the x-axis.

Solution: We take the derivative and do some algebra,

$$\frac{dy}{dx} = y^3 - \frac{1}{4y^3} \Rightarrow 1 + \left(\frac{dx}{dy}\right)^2 = 1 + y^6 - \frac{1}{2} + \frac{1}{16y^6} = y^6 + \frac{1}{2} + \frac{1}{16y^6} = \left(y^3 + \frac{1}{16y^3}\right)^2$$

Then the area is

$$A = 2\pi\int_1^2 y\left(y^3 + \frac{1}{16y^3}\right)dy = 2\pi\int_1^2\left(y^4 + \frac{1}{16y^2}\right)dy = 2\pi\left[\frac{1}{5}y^5 - \frac{1}{16y}\right]_1^2 = \frac{253}{20}\pi$$