## 10.6 Alternating Series

A series that has alternating signs, i.e. $\sum(-1)^n b_n$, is called an alternating series, which means we get cancelations. So, a series may diverge in absolute value, but converge conditionally.

**Theorem 1** (Alternating Series Test). *Consider the series $\sum(-1)^n b_n$, where $b_n > 0$, and suppose*

(i) $b_{n+1} \leq b_n$ *(i.e. $b_n$ are decreasing) for all $n > N \in \mathbb{N}$*
(ii) $\lim_{n\to\infty} b_n = 0$ *(i.e. the series $b_n$ converges to 0)*

*then the series converges.*

State whether the following converge or diverge, and state why:

Ex: Alternating harmonic series: $\sum_{n=1}^{\infty}(-1)^n/n$

**Solution:** First we take the limit $\lim_{n\to\infty} 1/n = 0\checkmark$. Now, we show that the $(n+1)^{\text{th}}$ term is smaller than the $n^{\text{th}}$ term: $1/(n+1) \leq 1/n\checkmark$. Therefore, by the alternating series test, $\sum_{n=1}^{\infty}(-1)^n/n$ converges.

Ex: $\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$.

**Solution:** Lets take the limit of the sequence, $\lim_{n\to\infty} \frac{3n}{4n-1} = \lim_{n\to\infty} \frac{3}{4-1/n} = 3/4 \neq 0$. By the $n^{\text{th}}$ term test, this diverges.

Ex: $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^3+1}$

**Solution:** Taking the limit gives us $\lim_{n\to\infty} \frac{n^2}{n^3+1} = \lim_{n\to\infty} \frac{1}{n+1/n^2} = 0\checkmark$. Furthermore, $\left(\frac{n(2-n^3)}{(n^3+1)^2}\right) < 0$ for $n^3 > 2\checkmark$. Therefore, by alternating series test, the series converges.

*Error Estimation:* This may or may not show up on the exam. If it does show up it will be a minor question, so know how to do this, but don't put too much effort into it.

The remainder of an alternating series is given by its next term; i.e. for $\sum(-1)^n b_n$, $|R_n| \leq b_{n+1}$. Lets do one example on this:

Ex: Approximate the sum of $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ to three decimal places.

**Brief Solution:** We see that $b_7 = 1/5040 < 1/5000 = 0.0002$, so $s_6$ (the sixth partial sum) is correct up to three decimal places, which is $s_6 \approx 0.368$.

Now we move to a definition that will be used a lot from now on.

**Definition 1.** The series $\sum a_n$ is absolutely convergent if $\sum |a_n|$ converges. Otherwise, if $\sum a_n$ converges, but $\sum |a_n|$ diverges, then $\sum a_n$ is said to be conditionally convergent.

State whether the following are absolutely convergent, conditionally convergent, or divergent.

Ex: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$.

**Solution:** Taking the absolute value gives us $\sum_{n=1}^{\infty} |(-1)^n/n^2| = \sum_{n=1}^{\infty} 1/n^2$. This converges by p-test since $p > 1$. Therefore, the original series is absolutely convergent.

Ex: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.

**Solution:** Taking the absolute value gives $\sum_{n=1}^{\infty} |(-1)^n/n| = \sum_{n=1}^{\infty} 1/n$. This diverges by p-test since $p = 1$. However, it converges by alternating series because the limit is $\lim_{n\to\infty} 1/n = 0\checkmark$ and $1/(n+1) \leq 1/n\checkmark$. So, this series converges conditionally.

**Theorem 2.** *If $\sum a_n$ converges absolutely, then it converges.*

Ex: Does $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ converge?

**Solution:** Lets look at the sum of the absolute values, $\sum_{n=1}^{\infty} \left|\frac{\cos n}{n^2}\right|$. Now, $\left|\frac{\cos n}{n^2}\right| \leq 1/n^2$. We know that $\sum_{n=1}^{\infty} 1/n^2$ converges by p-test since $p > 1$. Hence, $\sum_{n=1}^{\infty} \left|\frac{\cos n}{n^2}\right|$ also converges. Therefore, since the absolute value converges, the original series also converges.

Now lets do a few problems from the book that we did in class

7) First we take the limit,

$$\lim_{n\to\infty} 2^n/n^2 = \lim_{n\to\infty} \frac{e^{n\ln 2}\ln 2}{2n} = \lim_{n\to\infty} \frac{e^{n\ln 2}(\ln 2)^2}{2} = \infty \neq 0$$

So the series diverges by the nth term test.

11) First we take the limit,

$$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{1}{n} = 0\checkmark$$

Now, lets take the derivative to see if it is decreasing,

$$\left(\frac{\ln n}{n}\right)' = \frac{1-\ln n}{n^2} < 0 \text{ for } n \geq 3\checkmark$$

So, $\frac{\ln n}{n}$ is decreasing. Therefore the series converges by AST.

13) First we take the limit,

$$\lim_{n\to\infty} \frac{\sqrt{n}+1}{n+1} = \lim_{n\to\infty} \frac{1+1/\sqrt{n}}{\sqrt{n}+1/\sqrt{n}} = 0\checkmark$$

Now, we take the derivative to see if it's decreasing,

$$\left(\frac{\sqrt{n}+1}{n+1}\right)' = \frac{\frac{1}{2}n^{-1/2}(n+1)-(\sqrt{n}+1)}{(n+1)^2} = \frac{-\frac{1}{2}n^{1/2}+\frac{1}{2}n^{-1/2}-1}{(n+1)^2} < 0\checkmark$$

Therefore, the series converges by AST.

27) For this one we can use ratio test,

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)^2\left(\frac{2}{3}\right)^{\cancel{n+1}}}{n^2\cancel{\left(\frac{2}{3}\right)^{n}}}\right| = \lim_{n\to\infty} \frac{2}{3}\left(1+\frac{1}{n}\right)^2 = \frac{2}{3} < 1$$

Therefore, by the ratio test, the series converges absolutely.

35) Here we can use direct comparison on the absolute value: $0 \leq \frac{|\cos n\pi|}{n^{3/2}} \leq 1/n^{3/2}$. $\sum_{n=1}^{\infty} 1/n^{3/2}$ converges by p-test since $p > 1$, therefore $\sum_{n=1}^{\infty} \frac{|\cos n\pi|}{n^{3/2}}$ converge by DCT. Since the series of the absolute value converges, the original series converges absolutely.

41) There are a few ways to do this, but lets use partial sums here to go back to basics. Consider the series $\sum_{n=1}^{\infty} \sqrt{n+1}-\sqrt{n}$; i.e. the absolute value. Lets write down the first few terms,

$$\sum_{n=1}^{\infty} \sqrt{n+1}-\sqrt{n} = \sqrt{2}-1+\sqrt{3}-\sqrt{2}+\cdots+\sqrt{n}-\sqrt{n-1}+\sqrt{n+1}-\sqrt{n}+\cdots \Rightarrow S_n = \sqrt{n+1}-1 \Rightarrow \lim_{n\to\infty} S_n = \infty$$

So that series diverges. Now we test for conditional convergence. First take the limit

$$\lim_{n\to\infty} \sqrt{n+1}-\sqrt{n} = \lim_{n\to\infty} \frac{\cancel{n}+1\cancel{-n}}{\sqrt{n+1}+\sqrt{n}} = 0\checkmark$$

Then we show that it's decreasing,

$$\sqrt{n+2}-\sqrt{n+1} = \frac{1}{\sqrt{n+2}+\sqrt{n+1}} \leq \frac{1}{\sqrt{n+1}+\sqrt{n}} = \sqrt{n+1}-\sqrt{n}\checkmark$$

Therefore, the original series converges conditionally by AST.

51) Here we look at the remainder: $|R_4| \leq b_5 = (0.01)^5/5$.

## 10.7 Power Series

A power series is the variable analog of a geometric series, and looks like this: $\sum_{n=0}^{\infty} c_n x^n$. Notice that if we have the series $\sum_{n=0}^{\infty} x^n$, and we fix $x$, it is just a geometric series, so for $|x| < 1$, $\sum_{n=0}^{\infty} x^n = 1/(1-x)$. And we say that the series has a radius of convergence $R = 1$.

We can also have a power series of the form $\sum_{n=0}^{\infty} c_n (x-a)^n$, and this is called a power series centered at $x = a$. If $a = 0$, we are back to our usual power series, which we call a power series centered at $x = 0$, or simply a power series; i.e. if we don't mention what point it is centered around, it is by default centered at zero. Lets look at two simple examples,

Ex: Our $c_n$ need not be bounded. In this example we see what happens if we have an unbounded $c_n$. Consider $\sum_{n=0}^{\infty} n!x^n$. We use ratio test to see what our domain of convergence would be,

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} \left| \frac{(n+1)!x^{n+1}}{n!x^n} \right| = \lim_{n\to\infty} (n+1)|x| = \infty$$

except when $x = 0$, so the only point at which it converges is the origin.

Ex: Now, lets find the domain of convergence of something more interesting. Consider the series $\sum_{n=1}^{\infty} (x-1)^n/n$. Lets apply ratio test,

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} \left| \frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n} \right| = \lim_{n\to\infty} \frac{n}{n+3}|x-3| = \lim_{n\to\infty} \frac{1}{1+1/n}|x-3| = |x-3| < 3.$$

So, the series converges absolutely for $2 < x < 4$. Now, we test the end points. Notice when $x = 2$ this is an alternating series and will converge by the alternating series test. However, for $x = 4$, it is an harmonic series and does not converge by p-test since $p = 1$.

Now we shall present a few important theorems that don't have to be memorized, of course, but the ideas should be kept in mind.

**Theorem 3.** *Give $\sum_{n=0}^{\infty} c_n (x-a)^n$, we have three possibilities:*

(i) *The series converges at $x = a$,*
(ii) *The series converges for $|x-a| < R \in \mathbb{R}^+$,*
(iii) *The series converges for all $x$.*

**Theorem 4.** *If $\sum_{n=0}^{\infty} a_n x^n$ converges at $x = c \neq 0$, then it converges absolutely for all $|x| < |c|$, and if it diverges at $x = d$, then it diverges for all $|x| > d$.*

**Theorem 5.** *Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$, $B(x) = \sum_{n=0}^{\infty} b_n x^n$ converges absolutely for $|x| < R$, and $c_n = \sum_{n=0}^{n} a_k b_{n-k}$, then $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely to $A(x)B(x)$ for $|x| < R$.*

**Theorem 6.** *Suppose $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x| < R$, then for all continuous functions $f$ on $|f| < R$, $\sum_{n=0}^{\infty} a_n (f(x))^n$ also converges absolutely.*

**Theorem 7** (Term-by-term differentiation)**.** *If $\sum c_n (x-a)^n$ has a radius of convergence $R > 0$, it defines a function $f(x) = \sum c_n (x-a)^n$ on the interval $a - R < x < a + R$ and $f$ has derivatives of all orders on $(a-R, a+R)$.*

**Theorem 8** (Term-by-term integration)**.** *Suppose $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ converges for $a - R < x < a + R$; $R > 0$, then $\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$ converges in the same domain, and*

$$\int f(x)dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C. \tag{1}$$

Ex: For this example we demonstrate the usefulness of term-by-term integration. Consider,

$$\frac{1}{x} = \frac{1}{1-(1-x)} = \sum_{n=0}^{\infty}(1-x)^n = 1 + (1-x) + (1-x)^2 + \cdots$$

Integrating this term-by-term gives,

$$\ln x = \ln(1) + \sum_{n=0}^{\infty} \frac{(1-x)^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(1-x)^{n+1}}{n+1} = (1-x) + \frac{1}{2}(1-x)^2 + \frac{1}{3}(1-x)^3 + \cdots$$

And since the first series converges for $x \in (0,2)$ so does the second series.

Ex: Find the domain of convergence for $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$.

**Solution:** We first apply ratio test,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-3)^{n+1}x^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-3)^n x^n}\right| = \left|-3x\sqrt{\frac{n+1}{n+2}}\right| = 3|x|\sqrt{\frac{1+1/n}{1+2/n}}.$$

Taking the limit gives, $\lim_{n\to\infty}|a_{n+1}/a_n| = 3|x| < 1$, hence $|x| < 1/3 = R$. Now, if $x = -1/3$ the series diverges by p-test because $p < 1$, and if $x = 3$ the series converges by AST. So the domain of convergence is $-1/3 < x \leq 1/3$.

Ex: Find the domain of convergence of $\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$.

**Solution:** Again, we apply ratio test,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)(x+2)^{n+1}}{3^{n+2}} \cdot \frac{3^{n+1}}{n(x+2)^n}\right| = \frac{1+1/n}{3}|x+2|$$

Taking the limit gives, $\lim_{n\to\infty}|a_{n+1}/a_n| = \frac{1}{3}|x+2| < 1$, hence $|x+2| < 3 = R$. Now, if $x = -5$ or $x = 1$, the series diverges by the nth term test since it does not go to zero.

Here we do a few problems from the book that we did in class.

5) We can use either ratio test or root test for this one, with root test being a little quicker. Since we did root test in class, lets use ratio test here,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-2)^{\cancel{n+1}}}{10^{\cancel{n+1}}} \cdot \frac{\cancel{10^n}}{\cancel{(x-2)^n}}\right| = \frac{1}{10}|x-2| < 1 \Rightarrow |x-2| < 10.$$

So, the radius of convergence is $R = 10$ and the interval of absolute convergence is $-8 < x < 12$. Now we must also test the end points. For $x = -8$ the series becomes $\sum_{n=0}^{\infty}(-1)^n$ and for $x = 12$ the series becomes $\sum_{n=0}^{\infty} 1$, and both of these diverge by the nth term test, so the power series is nowhere conditionally convergent.

11) Again we use ratio test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \lim_{n\to\infty}\frac{1}{n+1}|x| = 0.$$

Since the limit goes to zero for all $x$, it has a radius of convergence $R = \infty$; i.e. the power series converges absolutely everywhere.

19) Ratio test yet again

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\sqrt{n+1}x^{n+1}}{3^{n+1}} \cdot \frac{3^n}{\sqrt{n}x^n}\right| = \lim_{n\to\infty}\frac{1}{3}\sqrt{1+\frac{1}{n}}|x| = \frac{1}{3}|x| < 1 \Rightarrow |x| < 3 = R.$$

So the interval of absolute convergence is $-3 < x < 3$. Now we must test the end points. If $x = \pm 3$, the series becomes $\sum_{n=0}^{\infty}(\pm 1)^n\sqrt{n}$, and this diverges by the nth term test, so the series isn't conditionally convergent anywhere.

21) Here we must split up the series, but in the end we get the same ratio test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\cancel{(x+1)^n}}{(x+1)^{\cancel{n+1}}}\right| = |x+1| < 1 = R$$

Then the interval of absolute convergence is $-2 < x < 0$. For both series and both end points the series diverge, so this is nowhere conditionally convergent.

23) Here we use root test, for once,

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\left(1+\frac{1}{n}\right)^n x^n\right|} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)|x| = |x| < 1 = R$$

So the interval of convergence is $-1 < x < 1$. At $x = \pm 1$ the series becomes $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n (\pm 1)^n$, which diverges by the nth term test because

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = \exp\left(\lim_{n\to\infty} n\ln(1+1/n)\right)$$

And

$$\lim_{n\to\infty} \frac{\ln(1+1/n)}{1/n} = \lim_{n\to\infty} \frac{(1/n)'/(1+1/n)}{(1/n)'} = \lim_{n\to\infty} \frac{1}{1+1/n} = 1 \Rightarrow \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e \neq 0$$

So the series is nowhere conditionally convergent.

27) We use ratio test,

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x+2)^{n+1}}{(n+1)2^{n+1}} \cdot \frac{n2^n}{(x+2)^n}\right| = \lim_{n\to\infty} \frac{n}{2n+2}|x+2| = \lim_{n\to\infty} \frac{1}{2+2/n}|x+2| = \frac{1}{2}|x+2| < 1 \Rightarrow |x+2| < 2 = R$$

Then the interval of absolute convergence is $-4 < x < 0$. If $x = -4$, the series becomes a harmonic series and diverges by p-test because $p = 1$. If $x = 0$, the series is now an alternating harmonic series which converges by AST. So the power series is conditionally convergent at $x = 0$.