10.6 Alternating Series and Absolute Convergence

A series that has alternating signs, i.e. $\sum(-1)^n b_n$, which means we get cancellations. We, of course, have a test for these situations.

Theorem 1. *Alternating series test: Consider the series $\sum(-1)^n b_n$, where $b_n > 0$, and*

(i) $b_{n+1} \leq b_n$ *(i.e. b_n are decreasing) for all $n > N$, where $N \in \mathbb{N}$*
(ii) $\lim_{n\to\infty} b_n = 0$ *(i.e. the series b_n converges to 0),*

then $\sum(-1)^n b_n$ converges.

State whether the following converge or diverge, and state the reasoning.

(1) Alternating harmonic series: $\sum_{n=1}^{\infty}(-1)^{n-1}/n$.
Solution: First we take the limit, $\lim_{n\to\infty} 1/n = 0$. Now, we show the $(n+1)^{\text{th}}$ term is smaller than the n^{th} term, $1/(n+1) \leq 1/n$ for all n. Therefore, by the alternating series test, $\sum_{n=1}^{\infty}(-1)^{n-1}/n$ converges.
(2) $\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$.
Solution: Lets take the limit of the sequence, $\lim_{n\to\infty} \frac{3n}{4n-1} = \lim_{n\to\infty} \frac{3}{4-1/n} = \frac{3}{4}$. Since this does not converge to 0, the series will diverge.
(3) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^3+1}$.
Solution: Taking the limit gives, $\lim_{n\to\infty} \frac{n^2}{n^3+1} = 0$.

Error estimation:
This may or may not show up on the exam. If it does show up it will be a minor question, so know how to do this, but don't put too much effort into it.

If $\sum(-1)^n b_n$ satisfies the alternating series test, then the remainder $|R_n| \leq b_{n+1}$.

Ex: Approximate the sum of $\sum_{n=0}^{\infty} \frac{(-1)^n}{n}$ to three decimal places.
Brief Solution: We see that $b_7 = 1/5040 < 1/5000 < .0002$, so s_6 (the sixth partial sum) is correct up to three decimal places, which is $s \approx s_6 = .368$.

Definition 1. The series $\sum a_n$ is <u>absolutely convergent</u> if $\sum |a_n|$ converges. Otherwise, if $\sum a_n$, but $\sum |a_n|$ diverges, then $\sum a_n$ is said to be <u>conditionally convergent</u>.

State whether the following are absolutely convergent, conditionally convergent, or divergent.

(1) $\sum_{n=1}^{\infty} (-1)^{n-1}/n^2$.
Taking the absolute value gives, $\sum_{n=1}^{\infty} |(-1)^{n-1}/n^2| = \sum_{n=1}^{\infty} 1/n^2$. We know this converges by p-series because $p > 1$. Therefore, $\sum_{n=1}^{\infty} (-1)^{n-1}/n^2$ is absolutely convergent.

(2) $\sum_{n=1}^{\infty} (-1)^{n-1}/n$.
Taking the absolute value gives, $\sum_{n=1}^{\infty} |(-1)^{n-1}/n| = \sum_{n=1}^{\infty} 1/n$. We know this diverges by p-series because $p = 1$. However, $\sum_{n=1}^{\infty} (-1)^{n-1}/n$ converges by the alternating series test, which we showed further up. Therefore, $\sum_{n=1}^{\infty} (-1)^{n-1}/n$ converges conditionally.

Theorem 2. *If $\sum a_n$ converges absolutely, then it converges.*

Ex: Does $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ converge?

Solution: Lets look at the sum of the absolute values, $\sum_{n=1}^{\infty} \left|\frac{\cos n}{n^2}\right|$.

Now, $\left|\frac{\cos n}{n^2}\right| \leq \frac{1}{n^2}$. We know that $\sum_{n=1}^{\infty} 1/n^2$ converges by p-series because $p > 1$. Hence, $\sum_{n=1}^{\infty} \left|\frac{\cos n}{n^2}\right|$ also converges. Therefore since $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ converges absolutely, it converges.